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Original Article

Conjugacy and Compatibility of Partial Arrays

S. Vijayachitra*[1] and K. Sasikala[2]

[1]Research Scholar, Department of Science and Humanity, Sathyabama University, Chennai, India
[2]Department of Mathematics, St. Joseph's College of Engineering, Chennai, India

***Corresponding Author**
S. Vijayachitra
Research Scholar,
Department of Science and Humanity,
Sathyabama University, Chennai, India
E-mail: vijayachitra@live.com

Abstract
Aldo de Luca (1999) developed a combinatorial method for the analysis of finite words for the study of biological molecules. Berstel and Boason introduced the partial words in the context of gene (or protein) comparision (Berstel and Boasson (1999)). The partial DNA arrays are used in developing an overall picture of how genes are regulated during hyphal development by studying the difference in gene expression between wild type and signaling mutants. In this paper we extend the concept of conjugate and compatibility of partial words to partial arrays.

Keywords:
Partial words, Partial arrays, conjugate and compatibility

1. Introduction

Research in combinatorics on words goes back a century. The stimulus for recent works on combinatorics is the study of biological sequences [1] such as DNA and protein that play an important role in molecular biology. Sequence comparison is one of the primitive operations in molecular biology. Alignment of two sequences is to place one sequence above the other [2] in order to make clear correspondence between similar letters or substrings of the sequences.

The compatibility relation [3]consider two arrays of same order with only few isolated insertions (or deletion). In some cases it allows insertion of letters which relate to errors or mismatches. A problem appears when the same gene is sequenced by two different labs that want to differentiate the gene expression. Also when the same long sequence is typed twice into the computer, errors appear in typing.

This paper studies a relation called K-compatibility where a number of insertions and deletions are allowed as well as K-mismatches. The conjugacy result which was proved for partial words is extended to partial arrays. The conjugacy problem of K-compatibility is discussed.

2. Preliminaries

In the first section we give a brief overview of partial words in the second section about partial arrays and in the third section about compatibility and conjugacy.

2.1 Partial words

Definition 2.1. *A partial word **u** of length **n** over A is a partial map **u**: {1, 2, . . . , n} → A. If 1 ≤ i ≤ n then i belongs to the domain of u (denoted by Domain(u)) in the case where u(i) is defined and i belongs to the set of holes of u (denoted by Hole(u)) otherwise.*

A word is a partial word over A with an empty set of holes.

Definition 2.2. Let u be a partial word of length n over A. The companion of u (denoted by $u_\Diamond$) is the map u: {1, 2, . . . , n} → A ∪ {◊} defined by

$$u_\Diamond(i) = \begin{cases} u(i) & if\ i \in Domain(u) \\ \Diamond & otherwise. \end{cases}$$

The symbol ◊ is viewed as a 'do not know' symbol. The bijectivity of the map u → $u_\Diamond$ allows us to define partial word concepts such as concatenation in a trivial way. The word $u_\Diamond$ = ba◊ab◊ is the companion of the partial word.

The length of the partial word is 6. D(u) = {1, 2, 4, 5}. H(u) = {3, 6}.

Definition 2.3. Two partial words u and v are called conjugate if there exist partial words x and y such that u ⊂ xy and v ⊂ yx.

Definition 2.4. Two partial words u and v are called K-conjugate if there exist non-negative integers K1, K2 whose sum is K and partial words x and y such that u ⊂K1 xy and v ⊂K2 yx.

2.2 Partial arrays

Definition 2.5. A partial array A of size (m, n) over Σ is a partial function $A_\Diamond : Z_+^2 \rightarrow \Sigma \cup \{\Diamond\}$ where Z_+ is the set of all positive integers. For 1 ≤ i ≤ m, 1 ≤ j ≤ n if A(i, j) is defined then we say that (i, j) belongs to the domain of A (denoted by (i, j) ∈ D(A)). Otherwise we say that (i, j) belongs to the set of holes of A (denoted by (i, j) ∈ H(A)).

An array over Σ is a partial array over Σ with an empty set of holes.

Definition 2.6. If A is a partial array of size (m, n) over Σ, then the companion of A (denoted by $A_\Diamond$) is the total function $A_\Diamond : Z_+^2 \rightarrow \Sigma \cup \{\Diamond\}$ defined by

$$A_\Diamond(i,j) = \begin{cases} A(i,j) & if\ (i,j) \in D(A) \\ \Diamond & otherwise \end{cases}$$

where $\Diamond \notin \Sigma$.

Example 2.1. The partial array $A = \begin{pmatrix} b & a & b \\ \Diamond & a & b \\ b & \Diamond & b \end{pmatrix}$ is the companion of a partial array A of size (3, 3) where D(A) = {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1), (3, 3)} and H(A) = {(2, 1), (3, 2)}.

Let $X = \begin{pmatrix} a_{m1} & \dots & a_{mn} \\ \vdots & & \vdots \\ a_{11} & \dots & a_{1n} \end{pmatrix}, Y = \begin{pmatrix} b_{m'1} & \dots & b_{m'n'} \\ \vdots & & \vdots \\ b_{11} & \dots & b_{1n'} \end{pmatrix}$ By column catenation we mean

$$X \oplus Y = \begin{pmatrix} a_{m1} & \dots & a_{mn} & b_{m'1} & \dots & b_{m'n'} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{11} & \dots & a_{1n} & b_{11} & \dots & b_{1n'} \end{pmatrix}$$

By row catenation we mean

$$X \ominus Y = \begin{pmatrix} a_{m1} & \dots & a_{mn} \\ \vdots & & \vdots \\ a_{11} & \dots & a_{1n} \\ b_{m'1} & \dots & b_{m'n'} \\ \vdots & & \vdots \\ b_{11} & \dots & b_{1n'} \end{pmatrix}$$

2.3 Compatibility and Conjugacy

If A and B are two partial arrays of equal size [4] then A is contained in B denoted by A ⊂ B if D(A) ⊆ D(B) and

A(i, j) = B(i, j) for all (i, j) ∈ D(A)

Definition 2.7. The partial arrays A and B are said to be compatible denoted by A ↑ B if there exists a partial array C such that A ⊂ C and B ⊂ C.

Definition 2.8. Two partial arrays A and B of same order are called conjugate if there exists partial arrays X and Y such that A ⊂ XY and B ⊂ Y X using row catenation or column catenation.

3. K-Compatibility in Partial Arrays

If A and B are two partial arrays of same order and K is non-negative integer then A is said to be K-contained in B denoted by A ⊂k B if D(A) ⊂ D(B) and there exists a subset E of D(A) of cardinality K called the error set such that

A(i, j) = B(i, j) for all (i, j) ∈ D(A) \ E

A(i, j) 6= B(i, j) for all (i, j) ∈ E

Definition 3.1. If A and B are two partial arrays of same order and K is a non-negative integer, then A and B are called K-compatible, denoted by A ↑k B if there exists a partial array Z and non-negative integers k_1, k_2 such that

- A ⊂k1 Z with error set E1
- B ⊂k2 Z with error set E2
- E1 ∩ E2 = _
- k1 + k2 = k

Example 3.1. $A = \begin{pmatrix} b & a & b \\ a & \Diamond & c \end{pmatrix}, B = \begin{pmatrix} a & b & a \\ a & \Diamond & c \end{pmatrix}$ then there exists a partial array $Z = \begin{pmatrix} a & b & b \\ a & \Diamond & c \end{pmatrix}$ with E1 = {(1, 1), (1, 2)}, E2 = {(1, 3)} and k1 = 2, k_2 = 1 ⇒ k = 3.

i.e., A ↑3 B.

Equivalently A and B are K-compatible if there exists a subset E of D(A) ∩ D(B) of cardinality K called the error set such that

- A(i, j) = B(i, j) ∀ (i, j) ∈ D(A) ∩ D(B) \ E
- A(i, j) 6= B(i, j) ∀ (i, j) ∈ E

If A and B are arrays then A ↑◦ B means A = B. We sometimes use the notation A ↑≤k B if the set E has the cardinality ≤ k.

4. Properties

Multiplication:

If A $\uparrow_{k_1}$ B and X $\uparrow_{k_2}$ Y then AX $\uparrow_{k_1+k_2}$ BY where A, B, X and Y are partial arrays and k1, k2 are non-negative integers, using column catenation.

Example 4.1 $A = \begin{pmatrix} \Diamond & a & a \\ b & b & \Diamond \\ a & b & a \end{pmatrix}, B = \begin{pmatrix} b & b & \Diamond \\ a & a & \Diamond \\ a & a & b \end{pmatrix}$

$X = \begin{pmatrix} b & \Diamond & a \\ a & b & a \\ a & \Diamond & b \end{pmatrix}, Y = \begin{pmatrix} a & b & b \\ b & a & b \\ \Diamond & \Diamond & a \end{pmatrix}$

AX ↑6+7 BY

Simplification:

If AX ↑k BY and order of A equal to order of B then A $\uparrow_{k_1}$ B and X $\uparrow_{k_2}$ Y for some k_1, k_2 satisfying $k_1 + k_2 = k$.

Example 4.2. $A = \begin{pmatrix} \Diamond & a & a \\ b & b & \Diamond \\ a & b & a \end{pmatrix}, B = \begin{pmatrix} b & b & \Diamond \\ a & \Diamond & a \\ b & a & b \end{pmatrix}$

$X = \begin{pmatrix} b & \Diamond \\ a & b \\ a & \Diamond \end{pmatrix}, Y = \begin{pmatrix} a \\ b \\ b \end{pmatrix}$

AX ↑8 BY ⇒ A ↑5 B and X ↑3 Y with 5 + 3 = 8.

Weakening:

If A ↑k B and Z ⊂ A then Z ↑≤k B.

Example 4.3. $A = \begin{pmatrix} \Diamond & a & a \\ b & b & \Diamond \\ a & b & a \end{pmatrix}, B = \begin{pmatrix} b & b & \Diamond \\ \Diamond & a & \Diamond \\ b & a & b \end{pmatrix},$

$Z = \begin{pmatrix} a & a \\ b & \Diamond \\ b & a \end{pmatrix}$

Z ↑≤7 B with k = 7.

Theorem 4.1. Let A and B be partial arrays of order a × b and a × c respectively. If there exists an array Z of order a × d and integers k_1, k_2, m and n such that A $\subset_{k_1}$ Z^m with error set E_1 and B $\subset_{k_2}$ Z^n with error set E2 then there exist integers p and q such that A^p $\uparrow_{\le k}$ B^q with

K = k(D(A)(a, |b|, p) ∩ E_2(a, |c|, q)) ∪ (D(B)(a, |c|, q) ∩ E_1(a, |b|, p))k

Moreover if E_1(a, |b|, n) ∩ E_2(a, |c|,m) = ϕ then A^p $\uparrow_k$ B^q.

Proof. Let A and B be partial arrays of a × b and a × c respectively. Let there exists an array z of order a × d such that by using column catenation

A $\subset_{k_1}$ Zm and B $\subset_{k_2}$ Z^n for some integers k_1, k_2, m and n. Let E_1 be the error set of cardinality k1 such that A(i, j) = Zm(i, j) for all (i, j) ∈ D(A)\E_1 and

A(i, j) 6= Zm(i, j) for all (i, j) ∈ E_1 and E_2 be the error set of cardinality k_2 such that B(i, j) = Z^n(i, j) for all (i, j) ∈ D(B) \ E_2 and B(i, j) 6= Z^n(i, j) for all (i, j) ∈ E_2. We have An $\subset_{n_{k1}}$ Zmn with error set E_1(a, |b|, n) of cardinality n_{k1} and Bm $\subset_{m_{k2}}$ Zmn with error set E2(a, |c|,m) of cardinality mk_2.

Let (1, 1) ≤ (i, j) ≤ (a, dmn) and Zmn(i, j) = a for some letter a. There are 4 possibilities.

Case (i)

If (i, j) 6∈ E_1(a, |b|, n), (i, j) 6∈ E2(a, |c|,m) then An(i, j) ∈ {_, a}, Bm(i, j) ∈ {_, a}. It does not give any error when we align An with Bm.

Case (ii)

If (i, j) 6∈ E_1(a, |b|, n), (i, j) ∈ E2(a, |c|,m) then An(i, j) ∈ {_, a} and Bm(i, j) = b for some b 6= a. It gives an error in the alignment of An with Bm only when An(i, j) = a or when (i, j) ∈ D(A)(a, |b|, n).

Case (iii)

If (i, j) ∈ E_1(a, |b|, n) and (i, j) ∈ E2(a, |c|,m) then Bm(i, j) ∈ {_, a} and An(i, j) = b for some b 6= a. It gives an error in the alignment of An with Bm only when Bm(i, j) = a or when (i, j) ∈ D(B)(a, |c|,m).

Case (iv)

If (i, j) ∈ E_1(a, |b|, n) and (i, j) ∈ E_2(a, |c|,m) then An(i, j) = b for some b 6= a and Bm(i, j) = c for some c 6= a. It gives an error in the alignment of An with Bm only when b 6= c.

Therefore if E_1(a, |b|, n) ∩ E_2(a, |c|,m) = _ then An ↑k Bm with k = k(D(a)(a, |b|, n) ∩ E2(a, |c|,m)) ∪ (D(B)(a, |c|,m) ∩ E1(a, |b|, n)k and E1(a, |b|, n) ∩ E2(a, |c|,m) 6= _ then An ↑≤k Bm.

Example 4.4. A = $A = \begin{pmatrix} a & b & \Diamond \\ c & a & b \\ \Diamond & b & a \end{pmatrix}, B = \begin{pmatrix} a & c \\ c & b \\ \Diamond & b \end{pmatrix}$

We have A ⊂4 Z3 with error set

E_1 = {(1, 2), (2, 2), (2, 3), (3, 3)},

B ⊂2 Z2 with error set

E2 = {(1, 2), (2, 2)}

K = 6

- D(A) = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)}

D(B) = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 2)}

- D(A)(a, |b|, 2) = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (1, 4), (1, 5), (2, 4), (2, 5), (2, 6), (3, 5), (3, 6)}

D(B)(a, |c|, 3) = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (1, 6), (2, 5), (2, 6), (3, 6)}

- E1(a, |b|, 2) = {(1, 2), (2, 2), (2, 3), (3, 3), (1, 5), (2, 5), (2, 6), (3, 6)}

E2(a, |c|, 3) = {(1, 2), (2, 2), (1, 4), (2, 4), (1, 6), (2, 6)}

E1(a, |b|, 2) ∩ E2(a, |c|, 3) 6= _

K = k(D(A)(a, |b|, 2) ∩ E2(a, |c|, 3) ∪ (D(B)(a, |c|, 3) ∩ E1(a, |b|, 2)k

= k(((1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (1, 4), (1, 5), (2, 4), (2, 5),

(2, 6), (3, 5), (3, 6)) ∩ ((1, 2), (2, 2), (1, 4), (2, 4), (1, 6), (2, 6)))

∪ (((1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (1, 6), (2, 5), (2, 6), (3, 6)) ∩ ((1, 2), (2, 2), (2, 3), (3, 3), (1, 5), (2, 5), (2, 6), (3, 6)))k

= k(1, 2), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5)(2, 6), (3, 6)k

K = 9

A2 ↑≤9 B3 (A2 ↑6 B3)

5. Conjugacy of K-Compatibility in Partial Arrays

Two partial arrays A and B of same order are K-conjugate if there exist non-negative integers K1K2 whose sum is K and partial arrays X and Y such that A ⊂K1 XY and B ⊂K2 Y X with row or column catenation.

0-conjugacy on partial words is reflexive and symmetric 0-conjugacy on partial arrays with same order is trivially reflexive and symmetric but not transitive.

Example 5.1.

$$A = \begin{pmatrix} a & \lozenge & b \\ b & c & a \\ a & a & \lozenge \end{pmatrix}$$

$$B = \begin{pmatrix} b & c & \lozenge \\ a & a & \lozenge \\ a & c & b \end{pmatrix} \; and \; C = \begin{pmatrix} b & a & \lozenge \\ b & a & \lozenge \\ a & b & c \end{pmatrix}$$

By taking X = (a c b) and Y = _ $Y = \begin{pmatrix} b & c & a \\ a & a & \lozenge \end{pmatrix}$ we get A ⊂ XY and B ⊂ Y X showing that A and B are conjugate similarly by taking $X' = \begin{pmatrix} b & c & \lozenge \end{pmatrix}$ and Y ′ = (a c b) we get B ⊂ X′Y ′ and C ⊂ Y ′X′ showing that B and C are conjugate. But A and C are not conjugate.

i.e., conjugate relation is not transitive.

Theorem 5.1. Let A and B be non-empty partial arrays of same order. If A and B are K-conjugate then there exists a partial array Z such that AZ ↑≤k ZB.

Proof. Let A, B be two partial arrays of same order. Suppose A and B are K-conjugate then by definition there exist non-negative integers K_1, K_2 whose sum is K and partial arrays X and Y such that A ⊂K1 XY with error set E_1 and B $\subset_{K2}$ Y X with error set E_2 using row catenation or column catenation accordingly.

Then AX $\subset_{K1}$ XY X with error set E_1 and XB $\subset_{K2}$ XY X with error set E′ 2 = {(i + number of rows of X, j)/(i, j) ∈ E2} or E′ 2 = {(i, j + number of columns of X)/(i, j) ∈ E_2} according as row or column catenation and so for Z = X we have AZ $\uparrow_{\leq k}$ ZB.

Example 5.2. Given

$$A = \begin{pmatrix} a & \lozenge & b \\ b & c & a \\ a & a & \lozenge \end{pmatrix}$$

$$B = \begin{pmatrix} b & c & b \\ a & a & \lozenge \\ a & \lozenge & b \end{pmatrix}$$

There exist and X = (a ◊ b) and $Y = \begin{pmatrix} c & a & b \\ a & a & \lozenge \end{pmatrix}$

with A $\subset_3$ XY and B $\subset_2$ Y X, K = K_1 + K_2 = 5.

There exist Z =(a ◊ b) such that AZ ↑≤5 ZB.

6. Conclusion

Motivated by K-compatibility and K-conjugate problem of K-compatibility of partial words we define K-compatibility between partial arrays. We verify some properties and prove that given partial arrays A,B and integers p, q satisfying |A|p = |B|q we find K such that A^p ↑K B^q. Also there exist partial array Z such that AZ ↑≤k ZB.

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