On convergences of contractive maps in metric spaces

N. B. Okelo*1 and Robert Obogi Karieko2

1School of Mathematics and Actuarial Science, Jaramogi Oginga Odinga University of Science and Technology, P. O. Box 210-40601, Bondo-Kenya
2Department of Mathematics, Kisii University, P. O. Box 408-40200, Kisii-Kenya

*Corresponding Author
N. B. Okelo
School of Mathematics and Actuarial Science, Jaramogi Oginga Odinga University of Science and Technology, P. O. Box 210-40601, Bondo-Kenya
E-mail: nivaare@yahoo.com

Abstract
In this paper, we introduce a new class of contraction maps, called A – contractions in fuzzy metric space. Under different sufficient conditions, existence of common fixed point for a pair of maps, four maps and also for a sequence of maps will be established here. Also it is shown that A – contractions is more generalized than TS – Contraction, B – Contraction in FM-space. If two fuzzy metrics are given on a set X, which are related, a pair of self map can have common fixed point though the contractive condition with respect one fuzzy metric is given. Our result extends, generalized and fuzzifies several fixed point theorems with A – contractions on metric space. We give generalizations and convergences of these maps.

Keywords:
Fuzzy metric spaces, A – Contraction, TS – Contraction, B – Contraction, Fixed point theorems

1. Introduction
Ever since the concept of fuzzy sets was introduced by Zadeh [5] in 1965 to describe the situation in which data are imprecise or vague or uncertain. It has a wide range of application in the field of population dynamics, chaos control, computer programming, medicine, etc. Kramosil and Michalek [7] introduced the concept of fuzzy metric spaces (briefly, FM–spaces) in 1975, which opened an avenue for further development of analysis in such spaces.

The study of common fixed points with A – contractions is new and also interesting. Works along these lines have recently been initiated by M. Akram, A. A. Zafar, A. A. Siddiqui [6] in 2008 and by Bhenga Akinbo, Memudu O. Olatinwo And Alfred O. Bosede [4] in 2010. In this article we introduce a new class of contraction maps, called A – contractions in fuzzy metric space. Under different sufficient conditions, existence of common fixed point for a pair of maps, four maps and also for a sequence of maps will be established here. Also it is shown that A – contractions is more generalized than TS – Contraction, B – Contraction in FM-space. If two fuzzy metrics are given on a set X, which are related, a pair of self map can have common fixed point though the contractive condition with respect one fuzzy metric is given. Our result extends, generalized and fuzzifies several fixed point theorems with A – contractions on metric space.

2. Preliminaries
We quote some definition and statements of a few theorems which will be needed in the sequel.

Definition 1.1 [2] A binary operation * [0, 1] × [0, 1] → [0,1] is continuous t – norm if * satisfies the following conditions:
(i) * is commutative and associative,
(ii) * is continuous,
(iii) a * 1 = a ∀ a ∈ [0, 1],
(iv) a * b ≤ c * d whenever a ≤ c, b ≤ d and a, b, c, d ∈ [0,1].

Result 1.2 [3] (a) For any r1, r2 ∈ (0, 1) with r1 > r2, there exist r3, r4 ∈ (0, 1) such that r1 * r3 > r2.
(b) For any r5 ∈ (0, 1), there exist r6 ∈ (0, 1) such that r5 * r6 ≥ r5.

Definition 1.3 [1] The 3-tuple (X, μ, *) is called a non–Archimedean fuzzy metric space if X is an arbitrary non-empty set, μ is a non–Archimedean fuzzy metric space if X is an arbitrary non-empty set. A sequence {x_n} in fuzzy metric space is said to converge to x ∈ X if and only if lim n→∞ μ(x_n, x, t) = 1.

Remark 1.4 In fuzzy metric space X, μ(x, y, t) is non – decreasing for all x, y ∈ X and μ(x, y, t) ≥ μ(x, z, t) * μ(y, z, t) for all x, y, z ∈ X, t > 0.

Note that μ(x, y, t) can be thought of as the degree of nearness between x and y with respect to t.

Note that μ(x, y, t) can be thought of as the degree of nearness between x and y with respect to t.

Note that μ(x, y, t) can be thought of as the degree of nearness between x and y with respect to t.
A sequence \( \{x_n\}_n \) in fuzzy metric space is said to be a \textbf{Cauchy sequence} if and only if \( \lim_{n \to \infty} \mu(x_{n+p}, x_n) = 1 \).

A fuzzy metric space \( (X, \mu, *) \) is said to be \textbf{complete} if and only if every Cauchy sequence in \( X \) is convergent in \( X \).

Let \( R_+^3 \) denote the set of all non-negative real numbers and \( A \) be the set of all functions \( \alpha: R_+^3 \to R_+^3 \) satisfying

(i) \( \alpha \) is continuous on the set \( R_+^3 \).

(ii) \( ka \geq b \) for some \( k \in (0, 1) \) whenever \( a \geq a(a, a, b, b) \) or \( a \geq a(b, a, b) \) or \( a \geq a(b, b, a) \) for all \( a, b \in R_+^3 \).

\textbf{Definition 1.6} [9] Let \( (X, \mu, *) \) be fuzzy metric space and \( T: X \to X \). \( T \) is said to be TS-contractive mapping if there exists \( k \in (0, 1) \) such that

\[ k \mu(Tx, Ty, t) \geq \mu(x, y, t) \]

\( \forall \ t > 0 \).

\textbf{Definition 1.7} Let \( (X, \mu, *) \) be fuzzy metric space \( T: X \to X \). \( T \) is said to be \( B \)-contraction if there exists \( k \in (0, 1) \) such that

\[ k \mu(Tx, Ty, t) \geq \min \left\{ \mu(x, Tx, t), \mu(y, Ty, t) \right\} \]

\( \forall x, y \in X \) and \( t > 0 \).

\textbf{Definition 1.8} A self-map \( T \) on a non-Archimedean fuzzy metric space \( X \), is said to be \( A \)-contraction if it satisfies the condition:

\[ \mu(Tx, Ty, t) \geq \alpha \left( \mu(x, y, t), \mu(x, Tx, t), \mu(y, Ty, t) \right) \]

for all \( x, y \in X \) and some \( \alpha \in A \).

\section*{3. Convergences of Contractive maps}

\textbf{Theorem 3.1} Every TS-contraction is an A-contraction on non-Archimedean fuzzy metric space \( (X, \mu, *) \) where

\[ a * b = \min \left\{ a, b \right\} \quad \forall \ a, b \in [0, 1] \]

\textbf{Proof:} Assume that \( T \) is an A-contraction on non-Archimedean fuzzy metric space \( X \) is TS-contraction. Define \( \alpha: R_+^3 \to R_+^3 \) by

\[ \alpha(u, v, w) = \frac{1}{k} \min \left\{ v, w \right\} \quad \text{for all } u, v, w \in R_+^3 \]

with some fixed \( k \in (0, 1) \). Next we show that \( \alpha \in A \).

\( i \) \text{ Clearly } \alpha \text{ is continuous.}

\( ii \) \text{ If } u \geq \alpha(u,v,v) \text{ then } u \geq \frac{1}{k} \min \{v,v\} = \frac{v}{k}.

\text{Similarly if } u \geq \alpha(v,v,u) \text{ then } ku \geq v.

So we deduce that \( \alpha \in A \). Further, since \( T \) is a TS-contraction, there exists \( k \in (0, 1) \) such that

\[ k \mu(Tx, Ty, t) \geq \mu(x, y, t) \quad \text{for all } t > 0 \]

\Rightarrow \mu(Tx, Ty, t) \geq \mu(x, Ty, t) \geq \mu(x, x, t) \mu(Tx, Ty, t) \mu(Ty, Ty, t) \mu(x, Ty, t) \mu(x, Ty, t) \mu(x, x, t)

\Rightarrow \mu(x, Ty, t) \geq \frac{1}{k} \min \{\mu(x, x, t), \mu(Tx, Ty, t), \mu(Ty, Ty, t)\}

\Rightarrow \mu(x, Ty, t) \geq \alpha(\mu(x, x, t), \mu(x, x, t), \mu(Ty, Ty, t))

This completes the proof.

\textbf{Theorem 3.2} Every \( B \)-contraction is an \( A \)-contraction on non-Archimedean fuzzy metric space \( (X, \mu, *) \), where

\[ a * b = \min \left\{ a, b \right\} \quad \forall \ a, b \in [0, 1] \]

\textbf{Proof:} Assume that \( T \) on the non-Archimedean fuzzy metric space \( X \) is \( B \)-contraction. By \( \alpha(u, v, w) = \frac{1}{k} \min \{v, w\} \)

\( \forall u, v, w \in R_+^3 \) with some fixed \( k \in (0, 1) \).

From the proof of the above theorem, we see that \( \alpha \in A \).

Further, since \( T \) is a \( B \)-contraction, we get

\[ \mu(Tx, Ty, t) \geq \alpha \left( \mu(x, y, t), \mu(x, Tx, t), \mu(Ty, Ty, t) \right) \]

\( \forall x, y \in X \) and \( t > 0 \)

\Rightarrow \mu(Tx, Ty, t) \geq \alpha \left( \mu(x, y, t), \mu(Tx, t), \mu(Ty, Ty, t) \right)

This completes the proof.

\section*{4. Fixed point Theorems}

In this section, we give some results on fixed points of \( A \)-contractions.

Let \( F, G, S \) and \( T \) be self-maps of a fuzzy metric space \( X \) satisfying

\[ SX \subseteq FX: TX \subseteq GX \]

Then for any point \( x_0 \in X \), we can find points \( x_1, x_2, \ldots \) in \( X \), such that

\[ x_0 = Fx_1, T x_1 = Gx_2, S x_2 = F x_3, \ldots \]

Therefore, by induction, we can define a sequence \( \{y_n\}_n \) in \( X \) as

\[ y_n = S x_n = F x_{n+1}, \text{ when } n \text{ is even and } y_n = T x_n = G x_{n+1}, \text{ when } n \text{ is odd} \]

The following theorem establishes existence of coincidence and unique common fixed point of \( F, G, S \) and \( T \) where the union of the ranges of \( F \) and \( G \) is required to be complete. The set of coincidence points of \( T \) and \( F \) is denoted by \( C(T, F) \), and the set of natural numbers denoted by \( N \).

\textbf{Theorem 4.1} Let \( F, G, S \) and \( T \) be self-maps of a non-Archimedean fuzzy metric space \( X \) satisfying \( 1 \) and for all \( x, y \in X \)

\[ \mu(Sx, Ty) \geq \alpha(\mu(Gx, Ty), \mu(Gx, Sx), \mu(Fy, Ty), \mu(Fy, Sx)) \quad \text{for all } \alpha \in A \]

\text{where } \alpha \geq \alpha(\mu(x, x), \mu(x, x), \mu(y, y), \mu(y, y)) \quad \text{for all } \alpha \in A \.

\text{Suppose } FX \cup GX \text{ is a complete sub-space of } X \text{, then the sets } C(T, F) \text{ and } \{S, G\} \text{ are nonempty. Suppose further that } (T, F) \text{ and } (S, G) \text{ are weakly compatible, then } F, G, S, T \text{ have a unique common fixed point.}

\textbf{Proof:} Assuming \( n \in N \) is even, then

\[ \mu(y_n \cdot y_{n+1} \cdot t) = \mu(S x_n \cdot T x_{n+1} \cdot t) \]

\[ \geq \alpha(\mu(G x_n \cdot F x_{n+1} \cdot t), \mu(G x_n \cdot S x_n \cdot t), \mu(F x_{n+1} \cdot T x_{n+1} \cdot t)) \]

\[ = \alpha(\mu(y_{n-1} \cdot y_{n-1} \cdot t), \mu(y_{n-1} \cdot y_{n-1} \cdot t), \mu(y_{n-1} \cdot y_{n-1} \cdot t)) \]

\[ \Rightarrow k \mu(y_n \cdot y_{n+1} \cdot t) \geq \mu(y_{n-1} \cdot y_{n-1} \cdot t) \]

On the other hand, assuming \( n \in N \) is odd, then

\[ \mu(y_n \cdot y_{n+1} \cdot t) = \mu(T x_n \cdot S x_{n+1} \cdot t) \]

\[ \geq \alpha(\mu(G x_n \cdot F x_n \cdot t), \mu(G x_n \cdot S x_{n+1} \cdot t), \mu(F x_n \cdot T x_n \cdot t)) \]
Thus whether \( n \) is odd or even, we have
\[
\mu\left(y_{n \cdot n} + 1 \cdot t\right) \geq \frac{1}{k} \mu\left(y_{n-1 \cdot n} + 1 \cdot t\right)
\]
for some \( k \in (0, 1) \).

Inductively, we have
\[
\mu\left(y_{n \cdot n} + 1 \cdot t\right) \geq \frac{1}{k} \mu\left(y_{0 \cdot 0} \right)
\]
and
\[
\lim_{n \to \infty} \mu\left(y_{n \cdot n} + 1 \cdot t\right) = 1
\]
Observe that \( \{y_n\} \) is contained in \( F \cup G X \). We now verify that \( \{y_n\} \)

is Cauchy sequence.
\[
\mu\left(y_{n \cdot n} + p \cdot t\right) \geq \mu\left(y_{n \cdot n} + 1 \cdot t\right) \cdots \mu\left(y_{n \cdot n + p - 1 \cdot n} + 1 \cdot t\right)
\]
\[
\lim_{n \to \infty} \mu\left(y_{n \cdot n} + p \cdot t\right) = 1
\]
\[
\lim_{n \to \infty} \mu\left(y_{n \cdot n} + p \cdot t\right) = 1
\]

Therefore \( \{y_n\} \) is Cauchy and \( F \cup G X \) is complete, there exists a point \( p \in F \cup G X \) such that
\[
\lim_{n \to \infty} y_n = p.
\]
Without loss of generality, let \( p \in G X \). It means we can find a point \( q \in X \) such that
\[
p = G q.
\]
Putting \( x = q \), \( y = x m \), \( M \) is odd, into (2) yields
\[
\mu\left(S q, T x, m \cdot t\right) \geq \alpha\left(\mu\left(G q, F x, m \cdot t\right), \mu\left(G q, S x, m \cdot t\right), \mu\left(F x, T x, m \cdot t\right)\right)
\]
\[
\Rightarrow \mu\left(S q, y, m \cdot t\right) \geq \alpha\left(1, 1, \mu\left(p, S q, t\right), 1\right)
\]
\[
\Rightarrow k \mu\left(S q, p, t\right) \geq 1
\]
\[
\Rightarrow \mu\left(S q, p, t\right) = 1.
\]
Consequently, \( p = S q \).

From \( S X \subseteq F X \) we know that there exists a point \( u \in X \) such that
\[
F u = S q = p = G q.
\]
Choosing \( X = q \), \( y = u, (2) \) gives
\[
\mu\left(S q, T u, t\right) \geq \alpha\left(\mu\left(G q, F u, t\right), \mu\left(G q, S u, t\right), \mu\left(F u, T u, t\right)\right)
\]
\[
\Rightarrow \mu\left(p, T u, t\right) \geq \alpha\left(1, 1, \mu\left(p, T u, t\right)\right)
\]
\[
\Rightarrow k \mu\left(p, T u, t\right) \geq 1
\]
\[
\Rightarrow \mu\left(p, T u, t\right) = 1.
\]
Consequently, \( p = T u \).

Hence, \( F u = T u = p = S q = G q \). This proves the first part of the theorem.

Now suppose \( (T, F) \) and \( (S, G) \) are weakly compatible pairs, then
\( F \) and \( T \) commute at \( u \), and \( G \) and \( S \) commute at \( q \) so that
\( F p = F F u = F T u = T F u = T p \) and
\( S p = S S q = S G q = G p \) \( \cdots (3) \)
Now with \( X = p \), \( y = u, (2) \) and (3) yield
\[
\mu\left(S p, T u, t\right) \geq \alpha\left(\mu\left(G p, F u, t\right), \mu\left(G p, S u, t\right), \mu\left(F u, T u, t\right)\right)
\]
\[
\Rightarrow \mu\left(S p, p, t\right) \geq \alpha\left(1, 1, \mu\left(p, T u, t\right), 1\right)
\]
\[
\Rightarrow k \mu\left(S p, p, t\right) \geq 1 \Rightarrow \mu\left(S p, p, t\right) = 1
\]
\[
\Rightarrow p = S p = G p.
\]
In a similar way, letting \( x = y = p, (2) \) and (3) yield
\( p = T p = F p \).

Thus, \( S p = G p = p = F p = T p \).

Finally, we show that \( p \) is unique in \( X \).

Suppose \( Z \) is another common fixed point of the four maps. Then
\[
\mu\left(Z, T p, t\right) \geq \alpha\left(\mu\left(G Z, F p, t\right), \mu\left(G Z, S p, t\right), \mu\left(F p, T p, t\right)\right)
\]
\[
\Rightarrow \mu\left(Z, p, t\right) \geq \alpha\left(\mu\left(Z, p, t\right), 1, 1\right)
\]
\[
\Rightarrow \mu\left(Z, p, t\right) = 1 \Rightarrow Z = p.
\]
This completes the proof.

**Corollary 4.2** Let \( S \) and \( T \) be self-maps of a non-Archimedean fuzzy metric space \( X \), satisfying
\[
\mu\left(S x, T y, t\right) \geq \alpha\left(\mu\left(x, y, t\right), \mu\left(x, S x, t\right), \mu\left(y, T y, t\right)\right)
\]
where \( \alpha \in A \) and for all \( x, y \in X \). Then \( S \) and \( T \) have a unique common fixed point.

**Theorem 4.3** Let \( F, G, S \) and \( T \) be self-maps of a non-Archimedean fuzzy metric space \( X \), and let \( \{S_n\} \) and \( \{T_n\} \) be sequences on \( S \) and \( T \) satisfying
\[
S_n X \subseteq F X : T_n X \subseteq G X, n = 1, 2, \cdots\quad (4)
\]
and for all \( x, y \in X \),
\[
\mu\left(S_{i+1} x, T_{j+1} y, t\right) \geq \alpha\left(\mu\left(G_{i+1} x, F_{j+1} y, t\right), \mu\left(G_{i+1} x, S_{i+1} x, t\right), \mu\left(F_{j+1} y, T_{j+1} y, t\right)\right)\quad \cdots (5)
\]
where \( \alpha \in A \). Suppose \( F \cup G X \) is a complete subspace of \( X \), then for each \( n \in N \),

(i) the sets \( C\left(F, T_n\right) \) and \( C\left(G, S_n\right) \) are nonempty.

Further, if \( T_n \) commutes with \( F \) and \( S_n \) commutes with \( G \) at their coincidence points, then

(ii) \( F, G, S \) and \( T \) have a unique common fixed point.

**Proof:** For any arbitrary \( x_0 \in X \) and \( n = 0, 1, 2, \cdots \), following a similar argument as in the beginning of this section, we can define a sequence \( \{y'_n\}_n \) in \( X \) as \( y'_n = S_n x_n = F x_{n+1} \), when \( n \) is even and \( y'_n = T_n x_n = G x_{n+1} \), when \( n \) is odd, where
\( n = 0, 1, 2, \cdots \).

Now for each \( i = 1, 3, 5, \ldots \) and \( n = 2, 4, 6, \ldots \), from (5) we have
\[
k \mu\left(y'_{i-1}, y'_{i+1}, t\right) \geq \mu\left(y'_{i-1}, y'_{i}, t\right)
\]
and
\[
k \mu\left(y'_{j-1}, y'_{j+1}, t\right) \geq \mu\left(y'_{j-1}, y'_{j}, t\right)
\]
That is, \( k \mu\left(y'_{n-1}, y'_{n+1}, t\right) \geq \mu\left(y'_{n-1}, y'_{n}, t\right)\) for all \( n = 0, 1, 2, \cdots \).

By induction (as in the proof of Theorem 4.1) we have
\[
\mu\left(y'_{n-1}, y'_{n+1}, t\right) \geq \frac{1}{k^n} \mu\left(y'_{0}, y'_{1}, t\right)
\]
for some \( k \in (0, 1) \).

Consequently, \( \{y'_n\}_n \) is Cauchy in \( F X \cup G X \), a complete subspace of \( X \).

The rest of the proof is similar to the corresponding part of the proof of Theorem 4.1.

**Theorem 4.5** Let \( T \) be an \( A \)-Contracton on a complete non-Archimedean fuzzy metric space \( X \). Then \( T \) has a unique fixed point in
$X$ such that the sequence $\{T^nx_0\}$ converges to the fixed point, for
any $x_0 \in X$.

**Proof:** Fix $x_0 \in X$ and define the iterative sequence $\{x_n\}$ by

$$x_n = T^n x_0 \quad \text{(equivalently, } x_{n+1} = T x_n \text{)}$$

where $T^n$ stands for the map obtained by $n$-time composition of $T$ with $T$. Since $T$ is an $\alpha$-Contraction, $\exists \alpha \in A$ such that the definition 1 holds, i.e.,

$$\mu(Tx,Ty) \geq \alpha(\mu(x,y), \mu(x,Tx), \mu(y,Ty)) \quad \text{for all } x, y \in X.$$ 

Now, we have

$$\mu(x_n, x_{n+1}) = \mu(Tx_n, x_{n+1}) \geq \alpha(\mu(x_{n-1}, x_n), \mu(x_n, x_{n+1}), \mu(x_{n-1}, x_n)) \quad \text{(6)}$$

Continuing this way, we get

$$\mu(x_n, x_{n+1}) \geq \frac{1}{k^n} \mu(x_0, x_1) \quad \text{and} \quad \lim_{n \to \infty} \mu(x_n, x_{n+1}) = 1 \quad \text{(7)}$$

Therefore $\{x_n\}$ is Cauchy sequence in $X$. Since $X$ is complete, there exists $x \in X$ such that

$$x_n \to x \text{ as } n \to \infty.$$ 

Again, with $x = x'$ and $y = x_n$, the inequality (6) gives

$$\mu(Tx, x_{n+1}) = \mu(Tx', x_{n+1}) \geq \alpha(\mu(x', x_n), \mu(x_n, x_{n+1}), \mu(x, x_{n+1})) \quad \forall n \in N.$$ 

By allowing $n \to \infty$ and using the continuity of $\alpha$, we get

$$\mu(x', x') = \lim_{n \to \infty} \mu(x_n, x_{n+1}) = 1$$

$$\text{and} \quad \lim_{n \to \infty} \mu(Tx', x') = 1$$

Therefore $\{x_n\}$ is Cauchy sequence in the complete fuzzy metric space $X$, so it converges to $x' \in X$. Next,

$$\mu(x', x_{m+1}, x') = \mu(Tx', x_{m+1}) = \mu(x'_m, x') \geq \alpha(\mu(x'_m, x_{m+1}), \mu(x', x_{m+1}), \mu(x'_m, x'_m)) \quad \text{for all } m \in N.$$ 

$$\text{and} \quad \lim_{m \to \infty} \mu(x'_m, x_{m+1}) = 1 \quad \Rightarrow \mu(x', x') = 1$$

$$\text{and} \quad \lim_{m \to \infty} \mu(Tx', x') = 1.$$ 

For uniqueness of the fixed point $x'$, we suppose $T_x y = y$ for some $y \in X$ and for all $n \in N$.

Then by (7), we have

$$\mu(x, y) = \mu(T_x y, y)$$

Therefore $\{x_n\}$ is Cauchy sequence in the complete fuzzy metric space $X$.
\[ \geq \alpha \left( \mu(\lambda', y', t), \mu(\lambda', T_i \lambda', t) \right) = \alpha \left( \mu(\lambda', y', t), 1, 1 \right) \Rightarrow \lambda' = y. \]

**Theorem 4.7** Let \( X \) be a set with two non-Archemedian fuzzy metrics \( \mu \) and \( \vartheta \) satisfying the following conditions:

(i) \( \mu(x, y, t) \geq \vartheta(x, y, t) \) for all \( x, y \in X \).

(ii) \( X \) is complete with respect to \( \mu \).

(iii) \( S, T \) are self maps on \( X \), such that \( T \) is continuous with respect to \( \mu \) and

\[ \vartheta(Tx, Sy, t) \geq \alpha \left( \vartheta(x, y, t), \vartheta(x, Tx, t), \vartheta(y, Sy, t) \right) \]

for all \( x, y \in X \) and for some \( \alpha \in A \).

Then \( S \) and \( T \) have a unique common fixed point.

**Proof:** Take any \( x_0 \in X \). For each \( n \in N \), we define \( x_n = Sx_{n-1} \), when \( n \) is even and \( x_n = T_{n-1} \), when \( n \) is odd. Then, by inequality in the above condition (iii) we get

\[ \vartheta(x_1, x_2, t) = \vartheta(Tx_0, Sx_1, t) \]

\[ \geq \alpha \left( \vartheta(x_0, x_1, t), \vartheta(x_0, Tx_0, t), \vartheta(x_1, Sx_1, t) \right) \]

\[ \geq \alpha \left( \vartheta(x_0, x_1, t), \vartheta(x_0, x_1, t), \vartheta(x_1, x_2, t) \right) \]

\[ \Rightarrow k \vartheta(x_1, x_2, t) \geq \vartheta(x_0, x_1, t) \]

In general, for any \( n \in N \) we get (as in the proof of the of the previous theorem) that

\[ \vartheta(x_n, x_{n+1}, t) \geq \frac{1}{k} \vartheta(x_0, x_1, t) \]

\[ \Rightarrow \mu(x_n, x_{n+1}, t) \geq \vartheta(x_n, x_{n+1}, t) \geq \frac{1}{k} \vartheta(x_0, x_1, t). \]

(By the condition (ii) )

\[ \lim_{n \to \infty} \mu(x_n, x_{n+1}, t) = 1 \]

This implies that \{ \( x_n \) \} is a Cauchy sequence in \( X \) with respect to \( \mu \) and hence by condition (ii), we have

\[ \lim_{n \to \infty} \mu(x_n, x', t) = 1 \] for some \( x' \in X \).

Since \( T \) is given to be continuous with the respect to \( \mu \) we have

\[ \lim_{n \to \infty} \mu(x_{2n}, x', t) = \lim_{n \to \infty} \mu(Tx_{2n}, x', t) = \mu(Tx', x', t) \]

\[ \Rightarrow T' = T'. \]

Now, by condition (iii)

\[ \vartheta(x', Sx', t) = \vartheta(Tx', Sx', t) \]

\[ \geq \alpha \left( \vartheta(x', x', t), \vartheta(x', Tx', t), \vartheta(x', Sx', t) \right) \]

\[ = \alpha \left( 1, 1, \vartheta(x', Sx', t) \right) \]

\[ \Rightarrow Sx' = x'. \]

Thus \( x' \) is a common fixed point of \( S \) and \( T \).

For the uniqueness, let \( y \) be any common fixed point of \( S \) and \( T \) in \( x \).

Then by condition (iii)

\[ \vartheta(x', y', t) = \vartheta(Tx', Sy', t) \]

\[ \Rightarrow \vartheta(x', y', t) = \alpha \left( \vartheta(x', y', t), \vartheta(x', Tx', t), \vartheta(y, Sy, t) \right) \]

\[ = \alpha \left( 1, 1, \vartheta(x', Sx', t) \right) \]

\[ \Rightarrow \vartheta(x', y', t) = 1 \]

\[ \Rightarrow x = y. \]

This completes the proof.

**Reference**


