Journal of Advanced Mathematics 02[07] 2016
www.asdpub.com/index.php/jam

Original Article

Variation of Depth with Oblique emergence of Rays Separated by a Finite Angle for Virtual Image

S. N. Maitra*

Department of Mathematics, National Defence Academy, Khadakwasla, Pune-411 023, India

***Corresponding Author**
Prof. S. N. Maitra
Department of Mathematics,
National Defence Academy,
Khadakwasla, Pune-411 023, India
E-mail: soumenmaitra@yahoo.co.in

Abstract
Unlike in previous published paper, two different rays separated by a finite angle, emitting from a point source in a denser medium enter a rarer medium following the laws of incidence and refraction. Two refracted rays thus obtained, when produced backwards meet at a point which is a virtual image as observed obliquely from the rarer medium. The longitudinal and lateral shifts of the image with respect to the object are determined.

Keywords:
Refraction, Finite Angle

1. Introduction

In almost all text books on Geometrical Optics is discussed formation of virtual image due to intersection of a normally refracted ray and an oblique path of a ray produced backwards [2].

SL Dhiman [1] invokes a novel idea that two rays of light emerging from a point source in one medium are incident on the interface with another medium, having their angles of incidences differing by infinitely small angle and subsequently, two refracted rays, when produced backwards intersect at a point which is a virtual image of the point source, as viewed by an observer in the latter medium. In other words he considered two neighboring rays emergent from a point source with their passage from one medium to another medium by refraction to form a virtual image. He derived expressions for both sideways and longitudinal shifts of this virtual image which are found to be finite, though the angle between the two incident rays is infinitesimally small and are functions of the angle of incidence and the refractive index of the denser medium.

Nevertheless in the light of SL Dhiman's derivations[1] and illustrations of graph and tables, a question arises as to why we should embark upon the problem of determining the location of the virtual image formed by any two arbitrary rays emerging from a point source in one medium , not necessarily ,separated by a small angle (that tends to zero), which are incident and thereafter refracted in the other medium. In the present write-up we first find the virtual image and its depth from the interface of the two media, when generated by one normally incident and refracted ray coupled with any other oblique rays emanating from a source A and then for any two angles of incidence, not necessarily differing by a small angle, on the surface of separation between the two media. The subsequent refracted rays are produced backwards and intersect at a point A^1 which is a virtual image of point source A.

2. Formation of virtual image and its location

Let a ray AB of light from a point source A be incident in a denser medium of refractive index μ *at an angle* i of incidence at point B on the interface with the rarer medium and subsequently be refracted as ray BD at an angle r of refraction. Another ray AC from the point source is normally incident at point on the interface XX^1 and is refracted normally intersecting the former refracted ray BD produced backwards at point A^1 which is a virtual image of the point source A, as viewed from the rarer medium.

If d and d^1 be depths of the object A and and of its virtual image A^1 respectively,

$$d=CA=x\cot i \qquad (1)$$

$$d^1=CA^1=x\cot r \qquad (2)$$

$$CB=x \qquad (2.1)$$

By Snell's law,

$$\text{Sini}/\text{Sinr}=\frac{1}{\mu} \qquad (3)$$

which gives $\text{Cosr}=\sqrt{1-\mu^2 Sin^2 i} \qquad (4)$

Eliminating x between (2) and (3), one gets after using (3) and (4)

$$d^1=\frac{d}{\mu Cosi}\sqrt{1-\mu^2 Sin^2 i}$$

$$\text{or,}\quad d^1=\frac{d}{\mu}\sqrt{1+(tan^2 i)(1-\mu^2)} \qquad (5)$$

$$\text{If } i\to 0,\ d^1=\frac{d}{\mu} \qquad (6)$$

In other words, if the virtual image is formed by one normally 'incident and refracted ray' and another 'ray incident and refracted in almost normal direction', relation (6) holds good for the object placed in the denser medium and vice-versa when placed in the rarer medium.

Formation of the virtual image by two oblique rays

Let us first define 'oblique ray' in the present context .A ray of light emerging from a Point source placed in one medium is incident at a point on the surface of separation with another medium and then is refracted in the second medium. Thus the passage of the ray from one medium (either denser or rarer) to another medium (either rarer or denser) is referred to as an oblique ray in the present quest.

Two rays of light, emitting from a point source A in a denser medium of refractive index μ *are incident at points B and* B^1 of the interface with rarer medium, say, air at angles i and θ *of incidence and* are refracted at angles r and $\emptyset$ of refraction respectively($\theta > i$). *These* two refracted rays BC and B^1 C^1 when produced backwards meet at a point A^1 which is a virtual image of object A. Lines OA and O^1A^1 are normal to the interface XX^1 on which lie O and O^1 suggesting that longitudinal and lateral shifts of the virtual image A^1 from the object A are ($h-h^1$) and $OO^1=x$ respectively, where h and h^1 are depths of the object A and the virtual image A^1 respectively.

3. Computation of depths and lateral shifts of the virtual image:

By Geometry and Trigonometry, we have

$OA=h$ and $O^1A^1=h^1$ so that $OO^1=OB-O^1B=OB^1-O^1B^1$

$$OO^1=x=h\tan i-h^1\tan r=h\tan\theta-h^1\tan\emptyset \qquad (7)$$

$$\text{Or,}\ h^1(\tan\emptyset-\tan r)=h(\tan\theta-\tan i)$$

$$h^1\sin(\emptyset-r)/\cos\emptyset\cos r=h\sin(\theta-i)/\cos\theta\cos i$$

$$\text{or,}\ h^1=h\sin(\theta-i)\cos\emptyset\cos r/\{\sin(\emptyset-r)\cos\theta\cos i\} \qquad (8)$$

using which in the first part of (7) one gets x=h$[\frac{sini}{cosi} - \{\sin(\theta - i)\, cos\emptyset . sinr\}/\{ \sin(\emptyset - r) cos\theta cosi\}]$ (8.1)

=h$[(sini \sin(\emptyset - r) cos\theta - sinr \sin(\theta - i)) cos\emptyset) / (\sin(\emptyset - r) cos\theta cosi)]$

=$\frac{h}{\sin(\emptyset - r) cos\theta cosi}$ $[\{\sin(\theta + i) + \sin(i - \theta)\} sin(\emptyset - r) - \{sin(\emptyset + r)$

$- sin(\emptyset - r)\} \sin(\theta - i)]/2$

=$\frac{h}{cos\theta cosi}[\sin(\theta + i) + \sin(i - \theta) \frac{sin\emptyset cosr + cos\emptyset sinr}{sin\emptyset cosr - cos\emptyset sinr}]/2$

Or, By snell's law (11),

x= $\frac{h}{cos\theta cosi}[\sin(\theta + i) + \sin(i - \theta)(\frac{\sqrt{(1-\mu^2 sin^2 i) sin^2\theta} + \sqrt{(1-\mu^2 sin^2\theta) sin^2 i}}{\sqrt{(1-\mu^2 sin^2 i) sin^2\theta} - \sqrt{(1-\mu^2 sin^2\theta) sin^2 i}})]/2$

(9)

which gives the shift of the virtual image A[1] from the object A in terms of the depth of the object and angles of incidence.

To find the depth of the virtual image below the interface XX[1] we rewrite equation (7) as

h[1]=h$(\tan\theta - tani)/(\tan\emptyset - \tan r)$ (10)

which in consequence of Snell's law rewritten as

sini/sin r=sin$\theta / sin\emptyset = 1/\mu$ (11)

turns out to be

h[1]=h$(\tan\theta - tani)/(\frac{\mu sin\theta}{\sqrt{1-\mu^2 sin^2\theta}} - \frac{\mu sini}{\sqrt{1-\mu^2 sin^2 i}})$

Or,

h[1]=h$(\tan\theta - tani).\sqrt{1-\mu^2 sin^2 i}\sqrt{1-\mu^2 sin^2\theta}$ /[$\mu\{\sqrt{(1-\mu^2 sin^2 i) sin^2\theta} - \sqrt{(1-\mu^2 sin^2\theta) sin^2 i}\}]$ (12)

Point source placed in one medium on entering an adjacent medium can form its virtual image in the same medium (former) after undergoing refractions provided the angles of incidence on the surface of separation between the two media are less than the critical angle (i.e., do not lead to 'total reflection'). Hence there can be multiple virtual images of an object, formed by 'refraction' with speculation that only bright ones can be seen from the other medium.

4. Formation of virtual image by a pair of rays separated by a very small angle

In this section we by use of two equations as above re-establish SL Dhiman's[1] theory of finding the position of the virtual image due to two oblique emergent rays very close to each other. To achieve this we need to put $\theta = i + di$ and $\emptyset = r + dr$ in the preceding analysis and as such rewrite (8.1) as

x=htani$[1 - \frac{\sin(\theta - i) \cos\emptyset\, sinr}{\sin(\emptyset - r) \cos\theta\, sini}]$ (13)

in which we replace θ and $\emptyset$ by i + di and r + dr respectively and take the case di $\to$ 0, so that as

di $\to 0$, dr $\to 0$ and (13) becomes

x=htani$[1 - \frac{di}{dr} . \frac{cosr . sinr}{cosi . sini}]$ (14)

From pedagogic point of view $\frac{di}{dr}$ ie, the rate of change of the angle of incidence with respect to the angle of refraction appearing in (14) has to be found out by Snell's law (11) in a manner simpler than that used by SL Dhiman[1].

Taking logarithm of (11) and differentiating both sides with respect to r, one ge

Coti$\frac{di}{dr} - Cotr = 0$

Or,$\frac{di}{dr} = \frac{\tan i}{\tan r}$ (15)

Employing (15) in equation (14), we get

x=htani $(1 - \frac{tani . cosr . sinr}{tani . cosi . sini})$

=htani $(1 - \frac{1-\mu^2 sin^2 i}{cos^2 i})$

Or, x=h$(\mu^2 - 1) tan^2 i$ (16)

In the same way, equation (8) gives

h[1]=h$\frac{di}{dr} . \frac{\cos(r + dr) cosr}{\cos(i + di) cosi}$ (with di$\to o, dr \to 0$ and using (15))

h[1]=h$\frac{tani\, cos^2 r}{tanr\, cos^2 i} = \frac{h}{\mu}(\frac{1-\mu^2 sin^2 i}{cos^2 i})^{3/2}$ (using Snell's law)

Or,h[1]=$\frac{h}{\mu}[1 - (\mu^2 - 1) tan^2 i]^{3/2}$ (17)

Numerical example1.

With i =10º ,$\mu = 1.5$ *in consequence of* (16)*and* (17) *in view of Dhaman's* [1] idea, x =006h, h [1] =0.62h (18)

5. Discussion

Equations(16) and (17),h[1] being replaced by y impress upon us to find the locus of the virtual image with respect to a system of axes such that the X-axis along OB and Y-axis along OA are depicted, O being the origin. Eliminating i between them we get

$\frac{\mu^2 y^2}{h^2}$ =1 $-(\mu^2 - 1)^3 tan^6 i - 3(\mu^2 - 1) tan^2$i{1-3$(\mu^2 - 1) tan^2$i}

=1-$(\mu^2 - 1)^3\{\frac{x}{h(\mu^2-1)}\}^2-3(\mu^2 - 1)\frac{x}{h(\mu^2-1)}\}^{2/3}$\{1-$(\mu^2 - 1)(\frac{x}{h(\mu^2-1)})^{\frac{2}{3}}\}\}$

Or,y²=$\frac{1}{\mu^2}[h^2 - x^2(\mu^2$-1)-3$(\mu^2 - 1)^{1/3} h^{4/3} x^{2/3}\{ \frac{1-(\mu^2-1)^{\frac{1}{3}} x^{\frac{2}{3}}}{h^{\frac{2}{3}}}\}]$ (19)

Obviously the maximum depth of the virtual image occurs at x=0, when the incident angle i=0, i.e., the ray passes normally. The virtual image occurs at the same level as the object when y=0, i.e., from (16).

Tani=$\frac{1}{\sqrt{(\mu^2-1)}}$ (20)

Hence such virtual image occurs at a distance x' from the object given by

x'=h$\sqrt{(\mu^2 - 1)}$ (21)

From (16) and (17) the angle of incidence, for which the sideways shift and the depth of the virtual image are equal is such that

$\{\{1 - (\mu^2 - 1) tan^2i\}^{3/2}$=$\mu(\mu^2 - 1) tan^3$i

Or,1-$(\mu^2 - 1) tan^2$i=$\{\mu(\mu^2 - 1)^{\frac{2}{3}}\} tan^2 i$

$tan^2 i = \frac{1}{(\mu^2-1)^{\frac{2}{3}}} . \frac{1}{\mu^{2/3} + (\mu^2-1)^{1/3}}$ (22)

References

[1] SL Dhiman, Variation of virtual depth with oblique emergence of rays, Bulletin IAPT, Vol 3,No 4, (April 2011) PP 101-105.

[2] Francis. A.Jenkins, "Fundamentals of Optics", Mc. Graw Hill Book Company, New Delhi, 4th Edition (1981).

