Journal of Advanced Mathematics 03[02] 2017
www.asdpub.com/index.php/jam

Original Article

Velocity altitude range and path of an aircraft performing optimal turn: Complete analytical solution

S.N. Maitra*

Retd HOD Mathematics, National Defense Academy, Kadakwasla, Pune, India

*Corresponding Author

Prof. S.N. Maitra
Retd HOD Mathematics,
National Defense Academy, Kadakwasla,
Pune, India
E-mail: soumen_maitra@yahoo.co.in

Keywords:

Accelerating control,
Aircraft

Abstract

Werner Grimm and Markus Hans established two kinds of optimal turn rate viz., accelerating control and decelerating control for an aircraft to obtain a specified heading and speed in a minimum time. Either of the two controls or both with transition of one control to the other are required to use for this purpose. Unlike in the previous papers in the present feature the velocity, range, altitude and interestingly curvilinear path acquired by the aircraft in an arbitrary time in course of either optimal turning are determined in closed form. A change of the boundary condition to simplify the optimization technique is also suggested. Finally a few numerical examples are cited.

1. Introduction

The differential equations[1] governing the flight of an aircraft in the horizontal plane is

$(T-D)/m = \dot{V} = f_0(V) - Cu^2, \quad \dot{\chi} = u$

With $f_0(V) = a_0 - a_1V^2 - (a_2/V^2)$

$a_0 = T/m$, $a_1 = \rho SC_{D0}/2m$, $a_2 = 2kW^2/(m\rho s)$

$$C = 2km/(\rho s), \qquad u_s(V) = \sqrt{fo(V)/C} \tag{1}$$

$\dot{x} = V\cos\chi, \quad \dot{y} = V\sin\chi$

$$\text{with } \tan\mu = (Vu)/g, \quad n = 1/\cos\mu = (W/C_Lqs)^{-1} \tag{2}$$

$D = qs(C_{D0} + KC_L^2)$

where for the aircraft

C_{D0} = Zero-lift drag coefficient

C_L = Lift coefficient

D = drag

g = gravitational acceleration

k = factor in the drag polar

m = aircraft mass

q = dynamic pressure

S = reference wing area

T = thrust_s

u = turn rate $\dot{x}$

u_s = steady – state turn rate

V = speed at time t

W = weight (= mg) of the aircraft

μ= bank angle

ρ = air density

χ= heading angle

The optimal control problem of Werner Grimm[1] and Markus Hans[1] is that the aircraft is to perform a specified heading change in minimum time subject to given boundary values upon the speed

$t_f \rightarrow$ Min

Subject to

$$\chi(0) = 0, \ \chi(t_f) = \Delta\chi, \ V(0) = V_0, \quad V(t_f) = V_f \tag{3}$$

Where obviously 0< ΔX < π → a right turn and ΔX > π → a left turn. In their model[1]1the induced drag is neglected, i.e., $a_2 = 0$, in model[l]2 obviously $a_2 \neq 0$.

They obtained two types of optimal control for head turning rate

$$(\dot{\chi})\, opt = u_{\pm} = u^* \pm \sqrt{(u^*)^2 - \{u_sV\}^2} \tag{4}$$

With costate $u^* = -(1/\lambda^*) > 0$

In previously published paper the value of this costate was not determined explicitly. Using the optimal turn rate equation (4) and the differential equation (1) of the acceleration is obtained [1]

$$\dot{V}_{\pm} = \mp 2C(u_{\mp})(\sqrt{(u^*)^2 - \{u_sV\}^2} \tag{5}$$

The entire papar[1] is devoted to evaluating the given heading ΔX as a function F(u*) of the costate u* resorting to cumbersome integrations by use of equations (4),(5) and boundary conditions (3) taking into account switch of one optimal control turning to the other.

Werner Grimm[1] and Markus Hans[1] refrained from determining the velocity- time distribution, velocity- heading distribution, distance traveled etc., in course of optimal turn. In the present paper attempts have been made to deal with these aspects so as to obtain complete analytical solution to the aircraft performance with such controlled heading rate in conformity with boundary conditions(3).

2. Optimal velocity – time distribution

In this section is solved equation (5) that has not been done earlier[1], subject to boundary conditions (3). From (1) we have

$$u_s^2(V) = 1/C\,[a_0 - a_1V^2 - a_2/V^2] \tag{6}$$

Which is used in (5); the dot sign denotes derivative with respect to time t. Substituting (4) into (5) we get

$$\dot{V} = \mp 2C[u^* \sqrt{(u^*)^2 - u_s(V)\}^2} \pm \sqrt{(u^*)^2 - \{u_s(V)\}^2}\,] \tag{7}$$

Further simplifying, rationalizing and using (6) one gets after replacing $u^*$by λ for convenience suitable forms:

$$2Cdt = \mp \frac{dV}{\lambda\sqrt{\lambda^2 - u_s^2(V)} \pm \left(\lambda^2 - u_s^2(V)\right)}$$

Which mean either

$$2Cdt = -\frac{dV}{\lambda\sqrt{\lambda^2 - u_s^2(V)} + \left(\lambda^2 - u_s^2(V)\right)} \tag{8}$$

Or

$$2Cdt = +\frac{dV}{\lambda\sqrt{\lambda^2 - u_s^2(V)} - \left(\lambda^2 - u_s^2(V)\right)} \tag{9}$$

Respectively with decelerating and accelerating controls we can integrate either of (8) and (9) with respect to t from o to t_f and V from initial velocity V_0 to final velocity V_f which occur during the same optimal control of the heading turn. Thus let us integrate (8):

$$2Cdt = -\frac{(\lambda^2 - u_s^2(V)) - \lambda\sqrt{\lambda^2 - u_s^2(V)}}{(\lambda^2 - u_s^2(V))^2 - \lambda^2(\lambda^2 - u_s^2(V))}dV \quad \text{or}$$

$$= \frac{\lambda^2 - u_s^2(V) - \lambda\sqrt{\lambda^2 - u_s^2(V)}}{(\lambda^2 - u_s^2(V))u_s^2(V)}dV$$

$2t_{f = I_1 - \lambda\sqrt{C}I_2}$

$$\text{where } I_1 = \int_{V_0}^{V_f} \frac{dV}{a_0 - a_1V^2 - \frac{a_2}{V^2}} \tag{10}$$

$$= \int_{V_0}^{V_f} \frac{dV}{\sqrt{4a_1a_2+a_0-(\sqrt{a_1}V+\frac{\sqrt{a_2}}{V})^2}}$$

$$= \int_{V_0}^{V_f} \frac{dV}{-(\sqrt{a_1}V-\frac{\sqrt{a_2}}{V})^2-2\sqrt{a_1a_2}+a_0}$$

Let us put $z_1 = \sqrt{a_1}V+\sqrt{a_2}/V$ and $z_2 = \sqrt{a_1}V-\sqrt{a_2}/V$ (11)

so that $dz_1 = \left(\sqrt{a_1}-\frac{\sqrt{a_2}}{V^2}\right)dV$ and $dz_2 = \left(\sqrt{a_1}+\frac{\sqrt{a_2}}{V^2}\right)dV$

$$I_1 = \frac{1}{2\sqrt{a_1}}\left[\int_{z_1(0)}^{z_1(f)} \frac{dz_1}{c_1^2-z_1^2} + \int_{z_2(0)}^{z_2(f)} \frac{dz_2}{c_2^2-z_2^2}\right]$$

Where $c_1^2 = 2\sqrt{a_{1a2}}+a_0$ and, $c_2^2 = a_0-2\sqrt{a_{1a2}}-a_0$ so that $c_1^2-z_1^2=c_2^2-z_2^2$

Hence

$$I_1 = \frac{1}{2\sqrt{a_1}}\left[\frac{1}{2c_1}\log\frac{c_1+z_1}{c_1-z_1}+\frac{1}{2c_2}\log\frac{c_2+z_2}{c_2-z_2}\right]_0^f$$

$$I_1 = \frac{1}{4\sqrt{a_1}}\left[\frac{1}{c_1}\log\left\{\frac{(c_1+z_1(f))}{(c_1-z_1(f))}\frac{(c_1-z_1(0))}{(c_1+z_1(0))}\right\}+\frac{1}{c_2}\log\left[\frac{(c_2+z_2(f))}{(c_2-z_2(f))}\frac{(c_2-z_2(0))}{(c_2-z_2(0))}\right]\right] \quad (12)$$

Where

$z_1(f) = \sqrt{a_1}V_f + \frac{\sqrt{a_2}}{V_f})$, $z_1(0)=\sqrt{a_1}V_0 + \frac{\sqrt{a_2}}{V_0}$,

$z_2(0)= \sqrt{a_1}V_0 - \frac{\sqrt{a_2}}{V_0}$ and $z_2(f) = \sqrt{a_1}V_f - \frac{\sqrt{a_2}}{V_f})$ (13)

$$I_2 = \int_{V_0}^{V_f} \frac{dV}{(a_o-a_1V^2-\frac{a_2}{V^2})(\lambda^2C-a_o+a_1V^2+\frac{a_2}{V^2})^{1/2}}$$

which with the help of the same technique as earlier can be reduced to the form:

$$I_2 = \frac{1}{2\sqrt{a_1}}\left[\int_{z_1(0)}^{z_1(f)} \frac{dz_1}{(c_1^2-z_1^2)\sqrt{c_3^2+z_1^2}} + \int_{z_2(0)}^{z_2(f)} \frac{dz_2}{(c_2^2-z_2^2)\sqrt{c_4^2+z_2^2}}\right]$$

$$= \frac{1}{2\sqrt{a_1}}(f_1+f_2) \quad (14)$$

Where $c_3^2 = \lambda^2C-a_0-2\sqrt{a_1a_2}$, $c_4^2 = \lambda^2C-a_0+2\sqrt{a_1a_2}$ (15)

Obviously

$$f_1 = \int_{z_1(0)}^{z_1(f)} \frac{dz_1}{(c_1^2-z_1^2)\sqrt{c_3^2+z_1^2}} = \int_{z_1(0)}^{z_1(f)} \frac{dz_1}{z_1^3(\frac{c_1^2}{z_1^2}-1)\sqrt{\frac{c_3^2}{z_1^2}+1}}$$

Let us put $\frac{c_3^2}{z_1^2}+1 = s_1^2$ so that $\frac{-2c_3^2}{z_1^3}dz_1 = 2s_1ds_1$

Then

$$f_1 = -\int_{s_1(0)}^{s_1(f)} \frac{ds_1}{c_1^2s_1^2-(c_1^2+c_3^2)} = -\frac{1}{c_1^2}\int_{s_1(0)}^{s_1(f)} \frac{ds_1}{s_1^2-c_5^2}$$

$$= -\frac{1}{2c_1^2c_5}\log\left\{\left(\frac{s_1(f)-c_5}{s_1(f)+c_5}\right)\left(\frac{s_1(0)+c_5}{s_1(0)-c_5}\right)\right\} \quad (16)$$

$$f_2 = -\frac{1}{2c_1^2c_6}\log\left\{\left(\frac{s_2(f)-c_6}{s_2(f)+c_6}\right)\left(\frac{s_2(0)+c_6}{s_2(0)-c_6}\right)\right\}$$

Where

$$\frac{c_{3+i}^2}{z_i^2} = s_i^2 \text{ (i=1,2);} \quad c_5^2 = \frac{c_1^2+c_3^2}{c_1^2}, \quad c_6^2 = \frac{c_2^2+c_4^2}{c_2^2} \quad (17)$$

Ultimately the time taken for such travel with optimal turning rate is given by the equations from (4) to (16) in terms of initial and final velocities.

$$t_f = F(V_0,V_f) = \frac{1}{8\sqrt{a_1}}\left[\frac{1}{c_1}\log\left\{\left(\frac{c_1+z_1(f)}{c_1-z_1(f)}\right)\left(\frac{c_1-z_1(0)}{c_1+z_1(0)}\right)\right\}+\frac{1}{c_2}\log\left\{\left(\frac{c_2+z_2(f)}{c_2-z_2(f)}\right)\left(\frac{c_2-z_2(0)}{c_2+z_2(0)}\right)\right\}\right]$$

$$+\sqrt{c}\,\lambda\left[\frac{1}{c_1^2c_5}\log\left\{\frac{s_1(f)-c_5}{s_1(f)+c_5}\cdot\frac{s_1(0)+c_5}{s_1(0)-c_5}\right\}+\frac{1}{c_1^2c_6}\log\left\{\frac{s_2(f)-c_6}{s_2(f)+c_6}\cdot\frac{s_2(f)+c_6}{s_2(f)-c_6}\right\}\right]$$

along with

$$s_1^2(0) = \frac{c_3^2}{z_1^2(0)}+1,\; s_1^2(f) = \frac{c_3^2}{z_1^2(f)}+1$$

$$s_2^2(0) = \frac{c_4^2}{z_2^2(0)}+1, s_2^2(f) = \frac{c_4^2}{z_2^2(f)}+1 \quad (18)$$

Whereas $z_i(0)$ and $z_i(f)$ are in terms of V_0 and V_f (i =1, 2) respectively as per relation (7).

3. Optimal velocity turn distribution

Combining first part of (1) with (4) and (5) followed by replacement of u^* by λ we get a differential equation involving turn χ and velocity V which can be solved in closed form subject to boundary conditions (3):

$$\frac{d\chi}{dV} = \frac{u}{V} = \frac{dV}{\sqrt{\lambda^2-CVu_s^2}} \quad \text{(by use of (6))}$$

$$2\sqrt{C}\,d\chi = \frac{dV}{\sqrt{C\lambda^2-a_0+a_1V^2+\frac{a_2}{v^2}}} \quad (19)$$

$$2\sqrt{Ca_1}\,d\chi = \frac{VdV}{\left[\left(V^2+\frac{\lambda^2C-a_0}{2a_1}\right)^2+\left\{a_2-\left(\frac{\lambda^2C-a_0}{2a_1}\right)^2\right\}\right]^{1/2}}$$

Or

$$4\sqrt{Ca_1}\,\chi_f = \log\left[\frac{V_f^2+C_7+\sqrt{(V_f^2+C_7)^2+C_8}}{C_9}\right] \quad (20)$$

$$C_7 = \frac{\lambda^2C-a_0}{2a_1},\; C_s = \frac{4a_2a_1^2-(\lambda^2C-a_0)^2}{4a_1^2} \quad (21)$$

and $C_9 = V_0^2+C_7+\sqrt{(V_0^2+C_7)^2+C_8}$

From (20) V_f can be explicitly determined:

$$V_f^2+C_7+\sqrt{(V_f^2+C_7)^2+C_8} = C_9e^{4\sqrt{Ca_1}\chi_f}$$

$$-(V_f^2+C_7)^2+\{(V_f^2+C_8)^2+C_8\} = C_8 \quad (22)$$

Dividing the second by the first one of (22) we have

$$-(V_f^2+C_7)+\sqrt{(V_f^2+C_7)^2+C_8} = \frac{C_8}{C_9}C_9e^{-4\sqrt{Ca_1}\chi_f}$$

(23)

Subtracting (23) from (22) and simplifying one gets the velocity and optimal turn distribution, in other words velocity turn distribution with optimal turn rate as

$$V_f^2 = \frac{1}{2C_9}\left\{C_9e^{4\sqrt{Ca_1}\chi_f}-C_8e^{-4\sqrt{Ca_1}\chi_f}\right\}-C_7 \quad (24)$$

4. Velocity- horizontal distance during optimal turn

Employing (9.1) in the first and second of (2) and in consequence of (20) we obtain

$$\frac{dx}{dV} = \frac{V}{2}\left[\frac{1}{a_0-a_1V^2-\frac{a_2}{V^2}}-\frac{\lambda\sqrt{C}}{(a_0-a_1V^2-\frac{a_2}{V^2})(\lambda^2C-a_0+a_1V^2+\frac{a_2}{V^2})^{1/2}}\right]$$

$$\times\cos\left\{\frac{1}{4\sqrt{Ca_1}}\log\frac{V^2+C_7+\sqrt{(V^2+C_7)^2+C_8}}{C_9}\right\} \quad (25)$$

$$\frac{dy}{dV} = \frac{V}{2}\left[\frac{1}{a_0-a_1V^2-\frac{a_2}{V^2}}-\frac{\lambda\sqrt{C}}{(a_0-a_1V^2-\frac{a_2}{V^2})(\lambda^2C-a_0+a_1V^2+\frac{a_2}{V^2})^{1/2}}\right]$$

$$\times\cos\left\{\frac{1}{4\sqrt{Ca_1}}\log\frac{V^2+C_7+\sqrt{(V^2+C_7)^2+C_8}}{C_9}\right\} \quad (26)$$

Which can be integrated in closed form by some approximate process or in numerical method with given initial and final conditions.

5. Horizontal distance and optimal turn distribution

Dividing the first and second of (2) by the first of (1) and thereafter eliminating V by use of (4), (24) and the last of (1), we get

$$\frac{dx}{d\chi}=\frac{V\cos\chi}{u_{\pm}}=\frac{V\cos\chi}{\lambda\pm\sqrt{\lambda^2-u_s(V)}}\qquad (\lambda=u^*) \tag{27}$$

In view of relationships (19) to (24) it is clear that

$$C\lambda V^2-\left(a_0V^2-a_1V^4-a_2\right)=a_1\left\{\left(V^2+C_7\right)^2+C_8\right\}$$

$$a_0V^2-a_1V^4-a_2=\frac{a_1}{4C_9{}^2}\left\{C_9{}^2e^{4\sqrt{Ca_1}\chi}+C_8{}^2e^{-4\sqrt{Ca_1}\chi}\right\}^2 \tag{28}$$

$$-C\lambda\left[\frac{1}{2C_9}\left\{C_9{}^2e^{4\sqrt{Ca_1}\chi}-C_8e^{-4\sqrt{Ca_1}\chi}\right\}-C_7\right]$$

because of (24), which

$$V^2=\frac{1}{2C_9}\left\{C_9{}^2e^{4\sqrt{Ca_1}\chi}-C_8{}^2e^{-4\sqrt{Ca_1}\chi}\right\}-C_7 \tag{29}$$

Substituting (28) and (29) into (27) we have the right hand side of (27) as a function of turn χ i.e.,

$$\frac{dx}{d\chi}=F(\chi)\cos\chi \tag{30}$$

and

$$\frac{dy}{d\chi}=F(\chi)\sin\chi \tag{31}$$

Which can be numerically integrated using the initial and final conditions to obtain horizontal distances (x,y) described reckoning turn χ during the optimal turning i. e., to acquire a head in minimum time.

6. Optimal curvileaner path and velocity distribution

If S be the distance traveled by the aircraft along the optimal path at time t, by use of (9.1) we get

$\frac{ds}{dt}$=v, v=$\frac{ds}{dV}\dot{V}$ so that

dS=

$$\frac{V}{2}\left[\frac{1}{a_0-a_1V^2-\frac{a_2}{V^2}}-\right.$$

$$\lambda_C a0-a1V2-a2V2\quad \lambda_2 c-a0+a1V2+a2V2^{1/2}\,dV$$

(32)

Hence in the light of the foregoing analysis the continuous optimal curved distance

$$S_f=k_1-\lambda\sqrt{C}k_1 \tag{33}$$

Where $k_1=\frac{1}{2}\int_{V_0}^{V_f}\frac{V\,dV}{a_0-a_1V^2-\frac{a_2}{V^2}}$, $k_2=\frac{1}{2}\int_{V_0 2}^{V_f}\frac{V\,dV}{\left(a_0-a_1V^2-\frac{a_2}{V^2}\right)\left(a_0+a_1V^2+\frac{a_2}{V^2}\right)^{1/2}}$

With λ^2 C-a_0=$a_1$putting$V^2=w$ resulting in change of the limits$\sqrt{w_0}$and$\sqrt{w_f}$of integration$k_1$becomes

$$k_1=\frac{1}{4a_1}\int_{\sqrt{w_0}}^{\sqrt{w_f}}\frac{w\,dw}{\frac{a_0}{a_1}w-w^2-\frac{a_2}{a_1}}=\frac{1}{4a_1}\int_{\sqrt{w_0}}^{\sqrt{w_f}}\frac{w\,dw}{\frac{a_0^2}{4a_1^2}-\left(w-\frac{a_0}{2a_1}\right)^2}$$

Putting $\sqrt{\frac{a_0^2}{4a_1^2}-\frac{a_2}{a_1}}+\frac{a_0}{2a_1}=\alpha$ and $\sqrt{\frac{a_0^2}{4a_1^2}-\frac{a_2}{a_1}}-\frac{a_0}{2a_1}=\beta$

$$k_1=\frac{1}{4a_1}\int_{\sqrt{w_0}}^{\sqrt{w_f}}\frac{w\,dw}{(\alpha-w)(w+\beta)}=\frac{1}{4a_1}\int_{\sqrt{w_0}}^{\sqrt{w_f}}\frac{[\alpha(w+\beta)-\beta(\alpha-w)]dw}{(\alpha+\beta)(\alpha-w)(w+\beta)}$$

(Performing optimal control) $=\frac{1}{4a_{1(\alpha+\beta)}}[-\alpha\log(\alpha-w)-\beta\log(w+\beta)]$

where α+β=2$\sqrt{\frac{a_0^2}{4a_1^2}-\frac{a_2}{a_1}}$, $w_{0=}v_0^2$, $w_{f=}v_f^2$

$$K_2=\frac{1}{4a_1}\int_{V_0}^{V_f}\frac{d\left(a_1V^2+a_2\left(\frac{1}{V^2}\right)\right)-a_2d\left(\frac{1}{V^2}\right)}{\left(a_0-a_1V^2-\frac{a_2}{V^2}\right)\sqrt{\left(a_0+V^2+\frac{a_2}{V^2}\right)}}$$

$$=\frac{1}{4a_1}\left[-\int_{z_0}^{z_f}\frac{dz}{z(a+a_0-z)1/2}+2a_2\int_{V_0}^{V_f}\frac{dV}{V^3\left(a_0-a_1V^2-\frac{a_2}{V^2}\right)\sqrt{\left(a_0+a_1V^2+\frac{a_2}{V^2}\right)}}\right]$$

$$=\frac{1}{4a_1}(q_1+2a_2q_2)$$

where

$q_{1=}-\int_{V_0}^{V_f}\frac{dz}{z(a+a_0-z)1/2}$,(putting $z^2=a+a_0-z$)

$$=2\int_{Z_0}^{Z_f}\frac{dZ}{a+a_0-Z^2}=\frac{1}{\sqrt{a+a_0}}\log\left\{\frac{\sqrt{a+a_0}+Z_f}{\sqrt{a+a_0}-Z_f}\cdot\frac{\sqrt{a+a_0}-Z_f}{\sqrt{a+a_0}+Z_f}\right\}$$

where $z_i=a+a_1V_i^2+\frac{a_2}{V_i^2}$,(i=0,f)

$$q_{2=}\int_{V_i}^{V_f}\frac{2dV}{V^3\left(a_0-a_1V^2-\frac{a_2}{V^2}\right)\left(a+a_1V^2+\frac{a_2}{V^2}\right)^{1/2}}$$

$$=\int_{V_i}^{V_f}\frac{dV}{V(a_0V^2-a_1V^4-a_2)(a+a_1V^2)^{1/2}}\left[1-\frac{a_2}{2V^2(a+a_1V^2)}\right]$$

This is due to a_2 being a small factor contributing to the induced drag such that $\frac{a_2}{v^2}\ll(a+a_1v^2)$ and on binomial expansion the squares and other higher power the small terms are neglected.

Putting $a+a_1V^2=X,^2$ $V^2=\frac{X^2-a}{a_1}$ we get

$$q_2'=\int_{X_0}^{X_f}\frac{dX}{a_1(X^2-a)\left\{\left(\sqrt{\frac{a_0^2}{4a_1}-\frac{a_2}{a_1}}\right)^2-\left(V^2-\frac{a_0}{2a_1}\right)^2\right\}};\qquad \boldsymbol{V_i^2}=\frac{X_i^2-a}{a_1}\quad \text{(i=0,f)}$$

$$=\frac{1}{a_1}\int_{X_0}^{X_f}\frac{dX}{(x^2-a)\left(\sqrt{\frac{a_0^2}{4a_1}-\frac{a_2}{a_1}}-\frac{a_0}{2a_1}-\frac{a}{a_1}+\frac{X^2}{a_1}\right)\left(\sqrt{\frac{a_0^2}{4a_1}-\frac{a_2}{a_1}}+\frac{a_0}{2a_1}+\frac{a}{a_1}-\frac{X^2}{a_1}\right)}$$

(putting $\sqrt{\frac{a_0^2}{4}-a_1a_2}-\frac{a_0}{2}-a=-b$, $\sqrt{\frac{a_0^2}{4}-a_1a_2}+\frac{a_0}{2}+a=c$

$$=a_1\int_{X_0}^{X_f}\frac{dx}{(X^2-a)(X^2-b)(c-X^2)}$$

$$=a_1\left[\frac{1}{(a-b)(c-a)}\int_{X_0}^{X_f}\frac{dX}{X^2-a}+\frac{1}{(b-a)(c-b)}\int_{X_0}^{X_f}\frac{dX}{X^2-b}+\frac{1}{(c-a)(c-b)}\int_{X_0}^{X_f}\frac{dX}{c-X^2}\right]$$

$$q_2=\left[\frac{1}{(a-b)(c-a)\sqrt{a}}\log\left(\frac{X_f-\sqrt{a}}{X_f+\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)+\frac{1}{(b-a)(c-b)\sqrt{b}}\log\left(\frac{X_f-\sqrt{b}}{X_f+b}\right)\left(\frac{X_0+\sqrt{b}}{X_0-\sqrt{b}}\right)+\frac{1}{(c-a)(c-b)\sqrt{c}}\log\left(\frac{\sqrt{c}+X_f}{\sqrt{c}-X_f}\right)\left(\frac{\sqrt{c}-X_0}{\sqrt{c}+X_0}\right)\right]\frac{a_1}{2}$$

so that

$$q_2'=A's_1+B's_2+C's_3$$

where $X_{i=}a+a_1V_i^2$ (i=o,f)

$$q_2''=\int_{V_i}^{V_f}\frac{dV}{V^3(a_0V^2-a_1V^4-a_1)(a+a_1V^2)^{3/2}}$$

(Simplifying on the above lines)

$$=\int_{X_i}^{X_f}\frac{dX}{(X^2-a)^2(X^2-b)(c-X^2)X^2}$$

$$=\int_{X_i}^{X_f}\left[\frac{A}{(X^2-a)^2}+\frac{B}{(X^2-a)}+\frac{C}{(X^2-b)}+\frac{D}{(c-X^2)}+\frac{E}{X^2}\right]dX$$

$$=\frac{A}{2a}\left(\frac{X_i}{X_i^2-a}-\frac{X_f}{X_f^2-a}\right)+\left(B-\frac{A}{2a}\right)s_1+Cs_2+Ds_3+E\left(\frac{1}{X_i}-\frac{1}{X_f}\right)$$

where

$$A'=\frac{a_1}{2}\frac{1}{(a-b)(c-a)\sqrt{a}},\quad B'=\frac{a_1}{2}\frac{1}{(b-a)(c-a)\sqrt{b}}$$

$$C'=\frac{a_1}{2}\frac{1}{(c-b)(c-b)\sqrt{c}};\ \boldsymbol{s_1}=\log\left(\frac{X_f-\sqrt{a}}{X_f+\sqrt{a}}\cdot\frac{X_0+\sqrt{a}}{X_0-\sqrt{a}}\right) \tag{34}$$

$$\boldsymbol{s_2}=\log\left(\frac{X_f-\sqrt{b}}{X_f+\sqrt{b}}\cdot\frac{X_0+\sqrt{b}}{X_0-\sqrt{b}}\right);\ \boldsymbol{s_3}=\log\left(\frac{\sqrt{c}+X_f}{\sqrt{c}-X_f}\cdot\frac{\sqrt{c}-X_0}{\sqrt{c}+X_0}\right)$$

A=$\frac{1}{a(a-b)(c-a)}$, **B**=$\frac{1}{(c-a)^2(c-b)c}-\frac{1}{(b-a)^2(c-b)b}+\frac{1}{a^2bc}$

C'=$\frac{1}{(b-)^2(c-b)b}$, **D**=$\frac{1}{(c-a)^2(c-b)c}$, **E**=$\frac{1}{a^2bc}$

In the binomial expansion evolved in the above integral q_2 for the sake of higher accuracy if we require to include the square and higher power of the small term $\frac{1}{(a+a_1V^2)}$, we shall encounter therein an integral of the type $I_n=\int_{X_i}^{X_f}\frac{dX}{(X^2-a)}n$ which can be avaluaded by reduction process as

$$I_n=\frac{1}{2(n-1)a}\left[\frac{X_i}{(X_i^2-a)}n-1-\frac{X_f}{(X_i^2-a)}n-1-(2n-3)I_{n-1}\right] \tag{35}$$

7. Discussion And Conclusion

In practice the entire optimal trajectory[1]corresponds to either one of the control type u₊and u- called single phase extremal or a composite structure of both types called two phase extremal depending on initial velocity V_0 *and final velocity* V_f. In a single phase[1]extremal one of the two control types is exerted in attaining final velocity V_f starting with initial velocity V_0, which is in this design restricted with some new insights.

Equation[11] in Reference1 given the optimal control u_+, u_- in terms of the instantaneous velocity V. Now using (7) and [11]/[4], ie,eliminating u_s(V) we can get the optimal control, ie, optimal turn rate in terms of the longitudinal acceleration as

$$\dot{V}_\pm=\mp 2C(u\pm)\left\{\pm\left(u_\pm-\lambda\right)\right\}, \qquad \lambda=u_*$$

$-\dot{V}$=2C$\{u_\pm^2-(u_\pm)\lambda\}$

Or

$$u_\pm=\frac{1}{2}\left(\lambda\pm\sqrt{\lambda^2-2\frac{\dot{V}}{C}}\right) \tag{36}$$

The unknown constant λ (=u*) can be found out in a simple manner unlike in Ref1 by suitably choosing the initial condition (3), i.e. taking $\dot{\chi}$=u₀ at t=0 then (4) given on simplification

$$\lambda=\left[\frac{u_0^2+u_s^2(V_0)}{2u_0}\right] \tag{37}$$

and the procedure for finding the optimal control to perform a specified head change $\Delta\chi$ remains the same but subject to the initial and boundary conditions;

$\chi(0)=0$, $\chi(t_f)=\Delta\chi$, v(0)=v₀ , χ (0)=u₀

But unlike ref1 herein the final velocity V_f is not specified for $(t_f)_{min}$, Obviously then because of (36) optimal turn rate in consequence of (4) becomes

$$u_\pm=\frac{u_0^2+u_s^2(V_0)}{2u_0}\pm\sqrt{\left(\frac{u_0^2+u_s^2(V_0)}{2u_0}\right)^2-u_s^2(V)} \tag{38}$$

$$\frac{du_{\pm-}}{du_0}=\left(1\pm\frac{\lambda}{\sqrt{\lambda^2-u_0^2(V_0)}}\right)\frac{d\lambda}{du_0}=\frac{u_{\pm-}}{\sqrt{\lambda^2-u_s^2(V_0)}}\left(1-\frac{u_s^2(V_0)}{u_0^2}\right)=0$$

when u₀=u_s(v₀) and also

$$\frac{d^2u_\pm}{du_0^2}=\frac{du_\pm}{du_0}\frac{1}{\sqrt{\lambda^2-u_s^2(V_0)}}\left(1-\frac{u_s^2(V_0)}{u_0^2(V_0)}\right)$$

$$+u\left(1-\frac{u_s^2(V_0)}{u_0^2(V_0)}\right)\frac{d}{du_0}\left(\frac{1}{\sqrt{\lambda^2-u_s^2}}\right)$$

$$+\frac{u}{\sqrt{\lambda^2-u_s^2(v_0)}}\times\frac{2u_s^2(V_0)}{u_0^3}=\frac{2uu_s^2(V_0)}{u_0^3\sqrt{\lambda^2-u_s^2(V_0)}}$$

$$>0\ for\ u_0=u_s(V_0)$$

which implies that $u_\pm$exhibits minimum when the initial head turn rate u_0 is equal $u_s(V_0)$and is given by

$$(u_\pm)min=u_s(V_0)\pm\sqrt{u_s^2(V_0)-u_s^2(V)} \tag{39}$$

and the corresponding longitudinal acceleration in view of (7) turns out to be

$$\dot{V}_\pm=\pm 2C(u_\pm)min\sqrt{u_s^2(V_0)-u_s^2(V)} \tag{40}$$

Again this minimum head turn rate can be further minimized in the same way with respect to the initial velocity v₀, i.e. by minimizing u₀=u_s(V₀). Recalling (1), u_s(V₀) has a minimum when

$$V_0=\sqrt[4]{\frac{a_2}{a_1}} \tag{41}$$

$$u_s(V_0)_{min}=a_0-2\sqrt{a_1a_2} \tag{42}$$

Also see figure 1 and 2

Figure1: Fighter aircrafts taking turns

Figure2: Fighter plane in climbing flight, yet to perform a turn.

Hence employing (42) in (39) we have

$(u_+)_{min2}=a_0-2\sqrt{a_1a_2}\pm\sqrt{(a_0-2\sqrt{a_1a_2})2-u_s^2(V)}$

and can find the corresponding longitudinal acceleration. In order to cite numerical examples we consider the following data.

Initial turning rate $(\dot{\chi})_0$ =u₀= 0.05

Initial velocity= V₀= 400 m/second

Velocity at a certain time during optimal control of turn =V₁ =500m/second

g=10N/sec², k=.072, W=16,0000 N

m=16000 kg, S=42 m², T=32,000N

ρ=1.225 kg/m³

Numerical example 1

Herein are computed in the light of the forgoing analysis stationary turn rate and the value of co-state λ (on account of modified initial conditions)

$a_0=\frac{T}{m}=\frac{320000}{16000}=20\ m/sec^2$ (1 N=100 GS)

$$a_1V_0^2=\frac{\rho\,SC_{oD}V_0^2}{2m}=\frac{1.225\times 42\times .02\times(400)^2}{2\times 1600}=5.15$$

$$\frac{a_2}{V_0^2}=\frac{2kw^2}{m\,\rho\,sV_0^2}=\frac{2\times .072\times(160000)^2}{16000\times 1.225\times 42\times((400)^2)}=.28$$

C=$\frac{2km}{\rho\,s}$=$\frac{2\times 0.072\times(16000)}{1.225\times 42}$; $\frac{1}{C}=2.3\times 10^{-4}$

so that the stationary turn rate $u_s(V_0)$ with initial velocity V_0 :

$u_s^2(V_0) = \frac{1}{C}\left(a_0 - a_1V_0^2 - \frac{a_2}{V_0^2}\right)$=.0034

Or, $u_s(V_0) = .058$

The value of co-state is

$$\lambda = \frac{1}{2}\left(u_0 + \frac{u_s^2(V_0)}{u_0}\right) = \frac{1}{2}(0.5 + 0.0034 \times 20) = 0.06$$

Numerical Example 2

The stationary turn rate with velocity $V_1 = 500 m/sec$ is :

$$u_s^2(V_1) = \frac{\left(a_0 - a_1V_1^2 - \frac{a_2}{V_1^2}\right)}{C} = (20 - 8.2) \times 2.3 \times 10^{-4} = .0027$$

Optimal control turn rate with this velocity is

$u_\pm(V_1) = \lambda \pm \sqrt{\lambda^2 - u_s^2(V_1)}$ =.06$\pm\sqrt{(.06)^2 - .0027}$

=0.06$\pm\sqrt{0.0009}$ =.06$\pm$.03 $= .09$ *or* 0.03

Numerical Example 3

The acceleration due to the optimal turn rate with initial velocity V_0= 400 m/Sec or 500m/Sec is given by

$\dot{V}_\pm$=$\mp 2Cu \pm (V_0).\sqrt{\lambda^2 - u_s^2(V_0)}$

=$\mp 2 \times \frac{10^4}{2.3} \times \frac{1}{20} \times \sqrt{(.06)2 - 0.0034}$ =$\pm 6.1\ m/Sec^2$

Or

$\dot{V}_\pm = \mp 2 \times \frac{10^4}{2.3} \times \frac{9}{100} \times \frac{3}{100}$=$\mp 23.5\ metre/sec^2$

Reference

[1] Werner Grimm and Markus Hans, Time- optimal turn to a heading: An improved analytical Solution, *Journal of Guidance, Control and dynamics*, vol 21, No 6, Nov-Dec 1998, PP. 940-947.