Bipolar- Radon Transform

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Abstract
This paper presents the bipolar- Radon transforms (BRT) that map a function into its integrals over two types of circles. By using the bipolar coordinates, this transform reduces to the normal Radon transform Radon transform (RT). (BRT) can be applied for tomography. A direct example and the inversions are presented.

Keywords: Bipolar- Radon transforms, tomography

1. Introduction
The (normal) Radon transform [9-12] maps an unknown function \( U(\xi, \eta) \) into its integrals over the lines \( L(s, \alpha) : (\xi, \eta) : \xi \cos \alpha + \eta \sin \alpha = s \) in \( \xi \eta \)-plane, the inversion of the Radon transform is the solution of the reconstruction problem for \( U(\xi, \eta) \). These straight lines represent the beam paths. Here we assume that the beam paths are the cylinders with the circular cross sections, given by
\[
C_{s}(s, \alpha) = \{(x, y) : (x - a \cos \alpha)^2 + (y - a \sin \alpha)^2 = r^2(s)\}; 0 \leq \alpha < \pi, 2 \pi \leq \alpha < 3 \pi / 2
\]
and for each \( \alpha \in (0, \pi / 2) \) and \( \alpha \in (\pi, 3 \pi / 2) \) and for each \( \alpha \in (\pi, 3 \pi / 2) \)
\[
G_{s}(s, \alpha) = \{(x, y) : (x - a \cos \alpha)^2 + (y - a \sin \alpha)^2 = r^2(s)\}; 2 \pi \leq \alpha < 3 \pi / 2
\]
where
\[
\eta(s) = \frac{a}{\sin \alpha} \sqrt{r^2(s) - a^2}, \quad x(s) = \frac{a}{\cos \alpha} \sqrt{r^2(s) - a^2}
\]
In the following section, we describe (BRT) in details

1- The Bipolar- Radon Transform(BRT)
Given an integrable function \( f(x, y) \), for each \( s \in (0, \infty) \) and for each \( \alpha \in [0, \pi / 2] \) and \( \alpha \in [\pi, 3 \pi / 2] \) define:
\[
G_{s}(s, \alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy
\]
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for each \( s \in (0, \infty) \) and for each \( \alpha \in [0, \pi / 2] \) and \( \alpha \in [\pi, 3 \pi / 2] \)
\[
G_{s}(s, \alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy
\]
Using the bipolar transform
\[
B : (\xi, \eta) \rightarrow (\xi, \eta) = (x(\xi, \eta), y(\xi, \eta))
\]
\[
B \rightarrow \xi = \frac{\sinh \xi}{(\cosh \xi - \cos \eta)}, \quad y(\xi, \eta) = \frac{\sin \eta}{(\cosh \xi - \cos \eta)}
\]
it is easy to task to show that:
\[
B^{-1} : (x, y) \rightarrow B^{-1}(x, y) = (\xi(x, y), \eta(x, y))
\]
where
\[
\xi = \frac{2a}{x + y + a}, \quad \eta = \frac{2a}{x + y - a}
\]
The Jacobian of the bipolar transform is given by \( h_{x}, h_{y} \):
\[
h_{x} = h_{y} = \frac{a}{\cosh \xi - \cos \eta}
\]
Now, \( g_{s}(s, \alpha) \) and \( g_{s}(s, \alpha) \) can be calculated as follow,
\[
g_{s}(s, \alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy
\]
for each \( s \in (0, \infty) \) and for each \( \alpha \in [0, \pi / 2] \) and \( \alpha \in [\pi, 3 \pi / 2] \)
\[
g_{s}(s, \alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy
\]
for each \( s \in (0, \infty) \) and for each \( \alpha \in [0, \pi / 2] \) and \( \alpha \in [\pi, 3 \pi / 2] \)
\[
g_{s}(s, \alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy
\]
for each \( s \in (0, \infty) \) and for each \( \alpha \in [0, \pi / 2] \) and \( \alpha \in [\pi, 3 \pi / 2] \)
\[
g_{s}(s, \alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy
\]
for each \( s \in (0, \infty) \) and for each \( \alpha \in [0, \pi / 2] \) and \( \alpha \in [\pi, 3 \pi / 2] \)
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g_{s}(s, \alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy
\]
for each \( s \in (0, \infty) \) and for each \( \alpha \in [0, \pi / 2] \) and \( \alpha \in [\pi, 3 \pi / 2] \)
\[
g_{s}(s, \alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy
\]
2- Examples

Remark: whenever $\alpha = 0$, then the values $s$ will be the values of $\zeta$, so in this example we will replace the notation "$s$" by "$\zeta$", and some specific value of $s$, say $S_0$, will be replaced by $\zeta_0$ and so on.

3- Consider the generalized function $f(x, y) = \delta(x - x_1), \delta(y - y_1)$, From Papoulis [13], $\delta'(x) = \sum \delta(x - x_i)$, where $x_i$ are the zeros of $\delta(x)$, then

\[
\delta(x(\zeta_0, \eta_0) - x_i) = \frac{\sinh \zeta_0 - \sinh \eta_0}{\cosh \zeta_0 - \cosh \eta_0} \cdot \eta_0 - \cos^{-1}[x, \cosh \zeta_0 - a \sinh \zeta_0]
\]

\[
\delta(y(\zeta_0, \eta_0) - y_i) = \frac{\cos \eta_0}{\cosh \eta_0 - \cosh \eta_0} \cdot [\eta_0 - \sinh^{-1}(\frac{x}{\sqrt{a^2 + y_i}}) + \sinh^{-1}(\frac{y}{\sqrt{a^2 + y_i}})]
\]

\[
\delta(x(\zeta_0, \eta_0) - y_i) = \frac{\cos \zeta_0}{\cosh \zeta_0 - \cosh \eta_0} \cdot [: \eta_0 - \cos^{-1}(\frac{x}{\sqrt{a^2 + y_i}}) - \cos^{-1}(\frac{y}{\sqrt{a^2 + y_i}})]
\]

\[
\delta(y(\zeta_0, \eta_0) - x_i) = \frac{\sin \eta_0}{\sinh \eta_0 - \sinh \eta_0} \cdot [\eta_0 - \sinh^{-1}(\frac{x}{\sqrt{a^2 + y_i}}) - \sinh^{-1}(\frac{y}{\sqrt{a^2 + y_i}})]
\]

To calculate $G_1(\zeta)$ from (2-8), we have to determine $\zeta_0(= \zeta^*_0)$ such that $\eta_0 = \eta_0$, in this case $G_1(\zeta) = 0$, and $G_1(\zeta) = 0$ for $\zeta \neq \zeta^*_0$. Similarly to calculate $G_2(\eta)$, we have to determine $\eta_0(= \eta^*_0)$ such that $\eta^*_0 = \eta_0$, in this case $G_2(\eta) = 0$, and $G_2(\eta) = 0$ for $\eta \neq \eta^*_0$. It is easy to see that

\[
\zeta^*_0 = \tan^{-1}\left[\frac{2ax}{x^2 + y^2 - a^2}\right], \eta^*_0 = \tan^{-1}\left[\frac{2ay}{x^2 + y^2 - a^2}\right]
\]

Then

\[
G_1(\zeta^*_0) = \begin{cases} c_1 d_1 [h(\zeta^*_0, \eta_0)], & \zeta^*_0 = \zeta^*_0 \\ 0, & \text{otherwise} \end{cases}
\]

where

\[
c_1 = \frac{\cosh \zeta^*_0 - \cos \eta_0}{\sinh \zeta^*_0 \cdot \sinh \eta_0}, \quad d_1 = \frac{\cos \eta_0 - \cosh \eta_0}{\cos \eta_0 \cdot \cosh \eta_0}
\]

Similarly, we have

\[
G_2(\eta^*_0) = \begin{cases} c_2 d_1 [h(\zeta^*_0, \eta^*_0)], & \eta^*_0 = \eta^*_0 \\ 0, & \text{otherwise} \end{cases}
\]

where

\[
c_2 = \frac{\cosh \zeta^*_0 - \cos \eta^*_0}{\sinh \zeta^*_0 \cdot \sinh \eta^*_0}, \quad d_2 = \frac{\cos \eta^*_0 - \cosh \eta^*_0}{\cos \eta^*_0 \cdot \cosh \eta^*_0}
\]

Then

For, $x \neq 0$ (except y-axis):

\[
G_1(\zeta^*_0) = \begin{cases} \frac{a(x \cosh \zeta^*_0 - \cos \eta_0)}{-a \sinh \zeta^*_0 \cdot \sinh \eta_0}, & \zeta^*_0 = \zeta^*_0 \\ \frac{-a \sinh \zeta^*_0 \cdot \cosh \eta_0}{a \sinh \zeta^*_0 \cdot \cosh \eta_0}, & \text{otherwise} \end{cases}
\]

\[
G_2(\eta^*_0) = \begin{cases} \frac{a(x \cosh \zeta^*_0 - \cos \eta^*_0)}{-a \sinh \zeta^*_0 \cdot \sinh \eta^*_0}, & \eta^*_0 = \eta^*_0 \\ \frac{-a \sinh \zeta^*_0 \cdot \cosh \eta^*_0}{a \sinh \zeta^*_0 \cdot \cosh \eta^*_0}, & \text{otherwise} \end{cases}
\]
3.2 Consider the case when there exists another source $\delta(x, y)$:

There are two possibilities:

First, if $(x, y, \eta) \neq (x, y, \eta, y) \in (\xi = \xi', \eta) \Rightarrow (x, y, \eta) \in (\eta = \eta')$; then $G_i(\xi', \eta')$ will change to another value $G_i(\xi', \eta') \neq G_i(\xi', \eta')$. In general, but $G_i(\xi', \eta')$ will change to $G_i(\xi', \eta') + G_i(\xi', \eta')$. It is easy to see that

$$G_i(\xi', \eta') = \frac{a(x, y, \eta)}{(x - a, \eta)}$$

Second, if $(x, y, \eta) \neq (x, y, \eta)$;

$(x, y, \eta) \in (\eta = \eta') \Rightarrow (x, y, \eta) \notin (\xi = \xi')$; then $G_i(\xi', \eta')$ will change to another value $G_i(\xi', \eta') = G_i(\xi', \eta') + G_x(\xi', \eta')$, in general, but $G_i(\xi', \eta')$ will change to $G_i(\xi', \eta') + G_i(\xi', \eta')$. It is easy to see that

$$G_i(\xi', \eta') = \frac{a(x, y, \eta)}{(x - a, \eta)}$$

Now, it is well known that there are many different methods such as the back projection method and spherical harmonics expansion to get the inversion formula of the Normal Radon transform, in other words there are many ways to reconstruct the unknown function $U(\xi, \eta)$ from its line integrals over the lines $L(s, \alpha) = \{ [\xi, \eta]: \xi \cos \alpha + \eta \sin \alpha = s \}$ in $\mathbb{R}^2$-plane. All the methods that work for Radon transform inversion can also work for bipolar-Radon transform.

In the next section, we get the inversion formula of (BRT) by using spherical harmonics expansion.

4-Inversion formula of the Bipolar-Radon Transform

For the purpose of this section let us rewrite the equations (2-8) and (2-9):

$$G_i(\xi', \eta') = \begin{cases} f(\xi, \eta) & \text{if } \xi = \xi', \eta = \eta' \\ f(\xi, \eta) & \text{otherwise} \end{cases}$$

Define $U(\xi, \eta)$, then

$$G_i(\xi', \eta') = \begin{cases} f(\xi, \eta) & \text{if } \xi = \xi', \eta = \eta' \\ f(\xi, \eta) & \text{otherwise} \end{cases}$$

Then [12]:

$$G_i(\xi', \eta') = \int_{S^{3n-1}} U(\xi, \eta) d\Omega$$

Define the transforms:

$$\gamma_1: (\xi, \eta) \rightarrow \gamma_1(\xi, \eta) := \left( \frac{\xi}{\eta}, \frac{\eta}{\xi} \right)$$

$$\gamma_2: (\xi, \eta) \rightarrow \gamma_2(\xi, \eta) := \left( \frac{\xi}{\eta}, \frac{\eta}{\xi} \right)$$

Then [12]:

$$G_i(\xi', \eta') = \int_{S^{3n-1}} U(\xi, \eta) d\Omega$$

By rotating the coordinate system,

$$x' = \cos \phi + \eta \sin \phi \Rightarrow (x', y') \Rightarrow \theta$$

$$y' = -\sin \phi \Rightarrow (x', y') \Rightarrow \theta$$

Then

$$\xi' = \tan^{-1} \left[ \frac{2\xi y'}{\xi'^2 + y'^2 + a^2} \right]$$

$$\eta' = \tan^{-1} \left[ \frac{2\eta y'}{\xi'^2 + y'^2 + a^2} \right]$$

Next, we get the inversion formula of (BRT) by using spherical harmonics expansion.
$$G_2(\eta_0) = G_2(\eta_0, \theta) = \int_{s', \theta' \in \varnothing} U \left( \frac{\eta_0}{a s_2}, \eta_0 \right) \frac{\eta_0}{a s_2} d \eta_0$$

(5-3)

$$\gamma_1': (\xi', \eta') \rightarrow \gamma_1'(\xi', \eta') = (\xi' / s_1, \eta' / s_1) = (\eta_{\theta_1}, \eta_{\theta_2}) = \alpha_1$$

$$\gamma_2': (\xi', \eta') \rightarrow \gamma_2'(\xi', \eta') = (\xi' / s_1, \eta' / s_1) = (\eta_{\theta_1}, \eta_{\theta_2}) = \alpha_2$$

Expanding \( U \cdot G \) and \( G_2 \) in spherical harmonics as follow:

$$U(\xi, \eta) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \int_{S^1} U_{2l}(\xi, \eta) Y_{2l}(\theta, \phi) d \theta d \phi$$

(5-4)

$$G_1(\xi_0) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \int_{S^1} g_{2l}(\xi_0) Y_{2l}(\theta)$$

(5-5)

$$G_2(\eta_0) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \int_{S^1} g_{2l}(\eta_0) Y_{2l}(\theta)$$

(5-6)

Apply (5-2) and (5-3) to the general term:

$$U(\xi, \eta) = U_{2l}(\xi_0, \eta_0) Y_{2l}(\theta, \phi)$$

where \( Y_l \) is the spherical harmonic of degree \( l \), defined in 2-dimensional as follow:

$$Y_{12}: S^1 \rightarrow \mathbb{C}$$

(5-7)

Then \( G_1(\xi_0) = \int_{S^1} U_{2l}(\xi_0, \eta_0) Y_{2l}(\theta_0, \phi_0) d \theta_0 d \phi_0 \)

Define

$$k_1(t) = \begin{cases} U_{2l}(\xi_0, \eta_0) Y_{2l}(\theta_0, \phi_0), & t > 0 \\ 0, & \text{otherwise} \end{cases}$$

Apply the Funk-Hecke theorem, then

$$G_1(\xi_0) = \int_{S^1} \frac{C_0}{r^l} \frac{g_{2l}(\xi_0)}{(\theta, \phi)} d \theta d \phi$$

Similarly for \( G_2(\eta_0) \) is the normalized Gegenbauer polynomial of degree \( l \)

Define

$$k_2(t) = \begin{cases} U_{2l}(\xi_0, \eta_0) Y_{2l}(\theta_0, \phi_0), & t > 0 \\ 0, & \text{otherwise} \end{cases}$$

Apply the Funk-Hecke theorem, then

$$G_2(\eta_0) = \int_{S^1} \frac{C_0}{r^l} \frac{g_{2l}(\eta_0)}{(\theta, \phi)} d \theta d \phi$$

Let \( r = \frac{\sqrt{t}}{s} \), then

$$G_2(\eta_0) = \frac{\sqrt{t}}{s} C_0 \frac{g_{2l}(\eta_0)}{(\theta, \phi)} d \theta d \phi$$

Finally, since \( U = (f \circ B) h \), see (5-1), then \( f \circ B = U \cdot h \). Then

$$f(x, y) = ((U \cdot h) \circ B^{-1})(x, y)$$

where \( h \) is defined in (2-4) and \( \cdot \) is defined in (2-5)

Algorithm

1. Get the \( x - y \) coordinates \((x_i, y_i), i = 1, \ldots, M \), on the boundary \( \Gamma_{\gamma_0} \) of interesting region \( D \) (the function support).

2. Get the corresponding values \((\xi(x_i, y_i), \eta(x_i, y_i)) \in \Gamma \)

3. Get the minimum value \( S_1(1) \) and the maximum value \( S_2(N_2 + 1) \) of the values \((\xi(x, y), \eta(x, y)), i = 1, \ldots, M \).

4. Make the partitions \( S_1(1) \) and \( S_2(N_2 + 1) \) to get the points \((S_1(i), S_2(j), j = 1, \ldots, N_2 + 1)

5. The image of these points under the map \((x(S_1(i), S_2(j)), y(S_1(i), S_2(j))) \) is points in the \( x - y \) plane that cover the function support.

6. Assume the unknown function is defined on these points \( f(x(S_1(i), S_2(j)), y(S_1(i), S_2(j))) \) and we want to find these values.

7. By rotating \( \xi - \eta \) plane, get:

$$S_1(i), S_2(j), \theta(k), i = 1, \ldots, N_1 + 1, j = 1, \ldots, N_2 + 1$$

8. By rotating \( x - y \) plane to be \((x - y) \in [0, \pi] \) for each \( \theta(k), k = 1, \ldots, N_1 + 1 \), the values of the function will be the same.

9. For each \( i = 1, \ldots, N_1 + 1, k = 1, \ldots, N_2 + 1 \),

$$G(S_1(i), S_2(j), \theta(k)) = \int_{(\xi, \eta) \in \Gamma_{\gamma_0}} f(\xi, \eta) |d\xi d\eta|$$

Similarly, for each \( j = 1, \ldots, N_2 + 1, k = 1, \ldots, N_2 + 1 \),

$$G(S_1(i), S_2(j), \theta(k)) = \int_{(\xi, \eta) \in \Gamma_{\gamma_0}} f(\xi, \eta) |d\xi d\eta|$$

10. Get the Fourier series:

$$G(S_1(i), \theta(k)) = \frac{1}{2\pi} \int_{0}^{2\pi} G(S_1(i), \theta(k)) d\theta(k)$$

$$G(S_2(j), \theta(k)) = \frac{1}{2\pi} \int_{0}^{2\pi} G(S_2(j), \theta(k)) d\theta(k)$$

where

$$g_{2l}(S_1(i)) = \frac{1}{2\pi} \int_{0}^{2\pi} G(S_1(i), \theta(k)) d\theta(k)$$

11. Get the Fourier transforms of \( S_1(i) \) and \( S_2(j) \) for each \( \theta(k) \):
Evaluate the integrals:

\[
\int_{-a}^{b} f(x) dx = \left\{ \begin{array}{ll}
1 & \text{on } D \\
0 & \text{otherwise}
\end{array} \right.
\]

Consider any circle centered at the point $\left( \xi_0, \eta_0 \right) = \left( \frac{1}{\sinh \xi_0}, 0 \right)$ with radius $\rho_0 = \frac{1}{\sinh \xi_0}$, $-\infty < \xi_0 < \infty$, so this circle is represented by the bipolar equation $\xi = \xi_0$.

By the properties of the bipolar coordinates, the circle $\xi = \xi_0$ intersects the unit disc for all $-\infty < \xi_0 < \infty$ (since $\alpha = 1$). By using the formula:

\[
\frac{1}{2} \int_{a+b \cos x}^{a+b \cos x} \frac{1}{ab} \ln \left| \frac{b^2 - a^2}{a^2 - b^2} \right| \tan \left( \frac{x}{2} \right) dx = a^2 > b^2
\]

The intersecting arc is given by

\[
G_1(\xi_0) = 2\xi_0 \tan^{-1} \left( \frac{\eta_0}{\xi_0} \right) \cos^{-1} \left( \sec h \xi_0 \right) - \frac{2}{\sinh \xi_0} \cos^{-1} \left( \sec h \xi_0 \right) = 2\xi_0 \tan^{-1} \left( \frac{\eta_0}{\xi_0} \right)
\]
Now, if the circle $\zeta = \zeta_0$ (which is setting on $xy -$ plane) is rotating by an angle $\theta$, then it will be setting on $x'y' -$ plane, with center $(x', y') = (\mu(\zeta_0), 0)$ and radius $r(\zeta_0) = \frac{1}{\sinh \zeta_0}$, the it will be defined by the equation $\zeta' = \zeta_0$, where

$$\zeta' = \tan^{-1} \left( \frac{2x'}{y'} \right).$$

Regarding to the domain $D = \{(x', y'): 0 \leq x'^2 + y'^2 \leq 1\}$, it remains has the equation $\eta' = \frac{\pi}{2}$ for the upper half $(x', y'): x'^2 + y'^2 = 1, y' > 0$ and $\eta' = \frac{3\pi}{2}$ for the lower half $(x', y'): x'^2 + y'^2 = 1, y' < 0$.

(Since the length is fixed by rotation).

Then

$$G_1(\zeta_0, \varphi) = \int \int [h(\zeta', \eta') d\eta'] = \frac{2\pi}{\sinh \zeta_0}.$$

Note that

$$[1] 0 \leq G_1(\zeta_0, \varphi) = G_1(\zeta_0) < \frac{2\pi}{\sinh \zeta_0}$$

$$[2] G_1(\zeta_0)$$ is a decreased function

$$[3] G_1(\zeta_0) = \frac{dG_1(\zeta)}{d\zeta} = g_{10}(\zeta) < 0$$

$$[4] U_{10}(r) = \frac{1}{\pi r^2} \left[ \zeta_0 - (r^2 - \zeta)^2 \right] C_{g_{10}}(\zeta, \eta, \varphi) d\zeta.$$

$C_{g_{10}}(\zeta) = \cos[I \cdot \cos^{-1}(x); |x| < 1; |C_{g_{10}}| = 1$.

On the other hand, Consider any circle centered at the point $(0, 0)$, with radius $r(\eta_0) = \frac{1}{\sin \eta_0}$, so this circle is represented by the bipolar equation $\eta = \eta_0; 0 < \eta_0 < \pi$. By using the formula:

$$\int \int \frac{1}{a + b \cos x} dx = \left[ \frac{2}{\sqrt{b^2 - a^2}} \tan^{-1} \frac{b - a}{\sqrt{b^2 - a^2}} \right] [\theta - |\theta|].$$

Then

$$G_1(\eta_0) = \int \int h(\zeta, \eta_0, \varphi) d\zeta = \frac{2}{\sin \eta_0} \tan^{-1} \frac{\sin \eta_0}{1 - \cos \eta_0}.$$

Similarly we did for $G_1(\xi_{\varphi}, \varphi)$, then

$$G_2(\eta_0, \varphi) = G_2(\eta_0) = \frac{2}{\sin \eta_0} \tan^{-1} \frac{\sin \eta_0}{1 - \cos \eta_0}.$$

For solving the inverse problem, given (6-1) and (6-2), then

$$U_{10}(r) = \frac{1}{\pi r^2} \left[ \eta_0 - (r^2 - \eta)^2 \right] G_{10}(\eta) d\eta.$$

Then

$$U(\xi, \eta) = \sum_{k=1}^{\infty} U_{10}(\xi_k, \eta) Y_{10}(\xi_k, \eta) \eta_0(\xi_k, \eta; \eta) : 0 \leq \eta \leq 2\pi.$$

Then

$$U(\xi, \eta) = U_{10}(\xi, \eta) \cos[0 - \frac{\eta}{\sqrt{\xi^2 + \eta^2}}] + U_{10}(\xi, \eta) \sin[0 - \frac{\eta}{\sqrt{\xi^2 + \eta^2}}]: 0 \leq \eta \leq 2\pi.$$

The conclusion is:

$$f(x, y) = (U / h) h^{-1}(x, y) = U(\zeta(x, y), \eta(x, y))$$

$$h(\zeta(x, y), \eta(x, y))$$

where

$$U(\zeta(x, y), \eta(x, y)) = U_{10}(\xi(x, y), \eta(x, y) + \eta^2(x, y)) =$$

$$= \frac{1}{\pi \sqrt{\xi^2 + \eta^2}} \int \frac{z^2}{\sqrt{\xi^2(\eta^2 + \eta^2(x, y))}} = 1 \left[ \frac{1}{2} \right] \sqrt{(x + 1)^2 + y^2}.$$
Figure 7: Plot of $f(x,y): \{(x,y) \in D: y = -\sqrt{(0.9)^2-x^2}, 0.1 \leq x \leq 0.9\}$

References