Original Article

On reflexivity, denseness and compactness of numerical radius attainable operators

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1. Introduction

We consider certain properties of operators. A lot of studies have been done on reflexivity, compactness and numerical radius attainability on Hilbert space operators [1-12] and the reference therein.

2. Preliminaries

2.1 Definition

An operator A ∈ B(H) attain its numerical radius if there are x0 ∈ H , f0 ∈ H* such that ∥x0∥ = ℓ∥f0∥ = f0(x0) = 1 and |f0(Ax0)|=|r(A)|, that is if the supremum defining r(A) is actually a maximum.

2.2 Lemma

Let each operator S ∈ M(A) be of rank one and attains its numerical radius. Then M(A) is reflexive.

Proof: For proof see [2].

3. Main Results

3.1 Theorem

Let M(A) be reflexive. Then it is Banach and for some y0 in Q(M(A)) the operator y* ⊗ y0* attaches a numerical radius for any y ∈ (M(A))*.

Proof.

Let M(A) be dense and non-reflexive. Suppose that every operator y* ⊗ y0* attains its numerical radius. By the Bishop-Phelps Theorem in [4] and the non re-reflective of M(A), we find (y* ⊗ y0*) ∈ Ψ(M(A))* which satisfies |y*| < 1 and ʃ X , and since y*0(y0*) = y*0(y0*) < 1 and since y*0(y0*) = 1, then y*0(y0*) = 1 and

\[ αy(y) = 1 \]

For some scalar β ≠ 0. By the Hahn-Banach Theorem, there ξ ∈ Q(M(A)) and t > 0

Such that \( \langle ξ, y \rangle = 0, \forall y \in M(A) \) and Re \( \langle ξ, y0* \rangle > t \). M(A) is dense, therefore in \( (M(A))^* \) the topology of strong convergence on \( M(A) \cup \{y0*\} \) is dense. Since Q(M(A)) is w∗-dense in \( Q(M(A))^* \), there exist a sequence \( \{y0^n\} \) in \( Q(M(A)) \) converges to \( \varphi \) in \( σ(M(A)), M(A) \cup \{y0]\). Then

\[ \{y0^n(y)\} \to 0, \forall y \in M(A) \]

And assume

\[ \Re y_n(y0*) \]

The set \( C = \bar{\{M(A)\}} \) and \( D = \bar{\{M(A)^*\}} \) (C) are considered as subsets of D. But the function \( f_n: \bar{\{M(A)\}} \to \mathbb{R} \) given by \( f_n(y)y^* = y^*(y0*)y0(y0*,(y0*,y0*)) \in \bar{\{M(A)^*\}} \). For each sequence \( \{g_n\} \) with 0 ≤ gn ≤ 1 such

\[ \sum_{n=1}^{∞}g_n(y,y*) = \Re y^*(\sum_{n=1}^{∞}g_ny_n)y0(y0), \forall (y,y*) \in \bar{\{M(A)^*\}} \].

We now get

\[ \sup_{y,y* \in \Psi(M(A))} \sup_{\varphi \in \Psi(M(A))} \Re y^*(\varphi) \Rightarrow \int_{\varphi \in \Psi(M(A))} \|\Re y_n(y0*)\| \leq \int_{\varphi \in \Psi(M(A))} \|\Re y_n(y0*)\| \leq \frac{1}{\beta} \]

and from (3) and (1), suppose \( x* \in (y0*) \), then \( \Re y^*(x^*) \leq \frac{1}{\beta} \).

Finally, from (4), (5) we get \( 0 \leq \frac{1}{\beta} \) but \( t > 0 \) which is a contradiction.

3.2 Theorem

Let \( Y \in M(A) \) be a rank one operator not attaining its numerical radius. Then \( M(A) \) can be renormed if it is infinite dimensional.

Proof.

Let \( M(A) \) to be reflexive and for normalized elements \( y_n \in B(M(A)^*\otimes M(A)) \), the equality \( \varphi(y_n \otimes y_0) = \|y_n \otimes y_0\| \) is true if \( s_n(y_n) = 1, \) since \( y_n \otimes y_0 \) is attained at \( y_nx_0 \in \bar{\{M(A)\}} \) [1, 2, 3, 4 and 5]. Now if \( y_n \otimes y_0 \) = 1 then we have \( s_n(y_n) = 1 = s_n^*(s) \) and commuting the elements \( s \) and \( s^* \) we obtain in \( \bar{\{M(A)\}} \) satisfying

\[ s_n(y_n) = 1 = s_n^*(s) \]

Let \( y_n^* \) be unique in the ball of \( M(A)^* \) and \( y_n^*(y_0) = 1. \) From the smoothness of \( y_n \) we obtain \( s^* = y_n^* \). Since \( (s, s^*) \in \bar{\{M(A)\}} \) \( x \) will be uniquely be determined by assuming that \( y_n^* \) is also smooth and so \( s = \lambda y_0 \) for some \( \lambda = 1 \) and \( (s, s^*) = (\lambda y_0, y_n^*) \). Using (1) again, \( s_n^*(y_n^*) = s_n^* = 1 \), and the smoothness of \( y_n \) gives us \( s_n^* = y_0 = s^* \). Finally, the couple \( (s, s^*) \) is \( (y_n, y_n^*) \). It is sufficient that \( s_n^* \otimes y_0 \) satisfies

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By inequality $2 \geq \|x_n + y_n\| \geq s_n(s_n + y_n)$ and (8), we have $\|x_n + y_n\| \to 2$. Similarly, if $s_n$ is a $w-$cluster point of $(s_n)$, (8) will also give us $s_n(s_n) = 1$. Conversely, if $s_n$ converges in the $w-$topology to an element $s_n$ in the unit ball and $\|s_n + y_n\| \to 2$, then there is a sequence of norm one functional $\{s_n\}$ so that the sequence $(s_n(s_n))$ and $(s_n(y_n))$ converges to $1$. By Bishop-Phelps-Bollobas Theorem [1, 2, 3, 4, 5] we assume that $s_n(s_n) = 1$ and so, we fix an element $s_n$ in the unit sphere of the dual so that $s_n(s_n) = 1$, and we have $\lim_{n \to \infty} s_n(s_n) = s_n(s_n) = 1$, and therefore $v(s_n \otimes y_n) \geq \sup_n s_n(s_n) y_n(y_n) \geq 1$, implying that the numerical radius of the operator is $1$.

3.3 Corollary

Let $M(A)$ be a Banach algebra. Then every operator in $M(A)$ can be perturbed by a normal operator to obtain an operator in $B(H)$. 

Proof. Suppose $X \in M(A)$ with $\|X\| = 1$ and $0 < \epsilon < \frac{1}{2}$ given. From [2, 3 and 4] two decreasing sequences of positive numbers, $(\alpha_n)$ and $(\beta_n)$ are chosen with the following conditions satisfied

$\sum_{n=1}^{\infty} (\alpha_n + 2\alpha_n^2) < \epsilon; \lim_{n \to \infty} a_n = \frac{1}{\alpha_n} \to 0$ (8)

We choose $\alpha_n = \frac{\epsilon}{32\gamma^2}$ for example, and $\delta = \alpha_n^2$. The sequence $X_n$ in $M(A)$ and $(\alpha_n f_n)$ in $\mathcal{L}(A)$ are constructed satisfying

$X_n = X_n, \quad \|f_n(X_n(a_n)) - v(X_n) - \delta_n\| \leq (\alpha_n^2 + 2\alpha_n) \leq 2, \forall \beta (9)$

$X_{n+1}(a) = X_n(a) + \alpha_n f_n(a) a_n + \alpha_n^2 f_n(a) a_n (a \in A)$ (10)

Where $|\lambda_n| = \frac{1}{\alpha_n^2}$ and $f_n(X_n(a_n)) = \delta_n f_n(X_n(a_n))$. It can be verified by induction that

$\|X_{n+1}\| \leq 1 + \sum_{n=1}^{\infty} (\alpha_n + 2\alpha_n^2) \leq 2, \forall \beta (11)$

By (12) and (7), the norm of the sequence $\{X_n\}$ converges to an operator $G$ in $M(A)$ satisfying

$\|G - X_n\| \leq \sum_{n=1}^{\infty} (\alpha_n + 2\alpha_n^2), \forall \beta, k (12)$

For all $n, k$, particularly $\|G - X\| \leq \epsilon$. With $X_n$, playing the role of $X, \delta_n = \delta_n, \alpha_n = \alpha_n, r = r = \epsilon_n + \epsilon_n + \epsilon_n, \epsilon_n \to 0$, so that the operator $X$ agrees with $X_n$ and we have

$\|G - X \| \leq \sum_{n=1}^{\infty} (\alpha_n + 2\alpha_n^2) + \sum_{n=1}^{\infty} \left(\sum_{n=1}^{\infty} (\alpha_n + 2\alpha_n^2) \right) + \sum_{n=1}^{\infty} (\alpha_n + 2\alpha_n^2) (14)$

Hence, by (7) and due to the fact that the sequence $\alpha_n \to 0$ and $\delta_n \to 0$, then $G \in B(H)$.

3.4 Theorem

Let $A \in B(H)$ be normal and $M(A)$ be compact and dense in $B(H)$. Then $A$ is compact.

Proof. Let $A \in B(H)$ and $M(A) \subseteq B(H)$. Suppose that $x_n$ is a strongly convergent sequence in $H$ then $A x_n$ is also a strongly convergent sequence in $M(A)$. As $A$ is normal then $M(A x_n) \to 0$ hence $M(A)$ is normal. But $M(A)$ is compact and dense. Then $A x_n \to 0$ for every strongly convergent sequence $(x_n)$ from $H$. Then we also have $A x_n \to 0$. Since $A$ is normal [4,7] then the operator $A^*$ is also normal. Since $x_n$ is a strongly convergent sequence in $H$ then $A^* A x_n \to 0$ and $A$ is closed. This implies that $A$ is compact.

References


